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ISOPERIMETRIC PROBLEM IN ANALYTIC DESIGN

I. A. Litovchenko

(Moscow)

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Original article submitted March 16, 1961

Analytic design [1] of an optimum controller is considered in the presence of an isoperimetric constraint and with saturation. Special points of the problem are illustrated by a simple example.

The problem of analytic design of controllers was solved in [2] under the condition that the modulus of deviation of the regulating member is bounded. But already in [3] our attention was drawn to the importance of the integral constraints imposed on the coordinates and/or on the speed of the controller. In the present work an attempt is made to clarify the role of such constraints in the structure of an optimum regulator such that the integral-square error of the system is minimized.

1. Isoperimetric Variational Problem

The motion of a certain class of closed-loop control systems can be described by the simultaneous differential equations

$$g_k \equiv \dot{\eta}_k - \left(\sum_\alpha b_{k\alpha}\eta_\alpha + m_k\xi\right) = 0, \tag{1}$$

where $b_{k\alpha}$ and m_k are known constants, η_k denote the phase coordinates and ξ the coordinate of the regulator, which is subjected to the isoperimetric constraint

$$\int_0^\infty \xi^2 dt = \varkappa_1. \tag{2}$$

A restriction such as (2) may apply to an arbitrarily narrow class of systems and would mean that very high demands are made on the system. A frequently considered [4] isoperimetric constraint including also the inequality sign (in our case this would read $\int_0^\infty \xi^2 dt \leqslant \varkappa_1$) presupposes the existence of "superfluous" ("extra") energy in the regulator, and finally also of "superfluous" weight, which in some cases would certainly be unwelcome.

It is assumed that η_k ($k = 1,2,\ldots,n$) and ξ belong to an open domain of the $(n + 1)$ dimensional Euclidean space. The obvious boundary conditions are imposed:

$$\eta_k(0) = \eta_{k0},$$
$$\eta_k(\infty) = 0 \quad (k = 1, \ldots, n). \tag{3}$$

The following variational problem of Lagrange type is proposed: to determine a system of functions minimizing the functional

$$I(\xi) = \int_0^\infty \sum_1^n a_k \eta_k^2 \, dt \quad (a_k > 0). \tag{4}$$

from the piecewise-smooth functions ξ, η_k ($k = 1,\ldots,n$) satisfying the simultaneous equations (1) as well as the isoperimetric constraint (2) and the boundary conditions (3).

The proposed problem can be solved by the methods of classical calculus of variations exactly as in [1]. All the arguments developed in [1] remain fully valid in this case also, as they are true for any coefficient of ξ^2 in the Lagrange function and in particular for the actual value of this coefficient; thus there is only a formal and not an essential difference between our problem and that of [1]. In [1] the Lagrange function H takes the form

$$H = \sum_1^n a_k \eta_k^2 + c\xi^2 + \sum_1^n \lambda_k g_k,$$

where $\underline{c}$ is a known weight constant (see [1], the formula (2.4)), and in the present problem

$$\tilde{H} = \sum_1^n a_k \eta_k^2 + \lambda\xi^2 + \sum_1^n \lambda_k g_k,$$

where λ is a constant Lagrange multiplier to be determined. Consequently, the result of the above is an isoperimetric problem which can easily be obtained simply by replacing $\underline{c}$ by λ in the final formulas of [1]. The multiplier λ is a root of the algebraic equation which one obtains from the isoperimetric restriction (2) by substituting ξ for the time function which can be determined from the Euler equations and from (1). The existence of at least one real root of this equation remains an open question, that is, whether the variational problem admits a solution at all. This in fact constitutes a special feature of our problem; generally speaking it leads to the apportioning in the phase space of a closed domain of initial values η_{k0} (k = 1,...,n) for which there exists a solution to the variational problem.

Similarly, for the isoperimetric constraint of the form $\int_0^\infty \dot{\xi}^2\, dt = \mu_1$ and the optimized functional $N(\xi) = \int_0^\infty (\Sigma a_k \eta_k^2 + c\,\xi^2) dt$, the reasoning given in section 4 of [2] remains also valid. The final formulas are identical up to constant multipliers. For instance, in [2] the equation of ξ is

$$2\ddot{\xi} = 2c\xi - \sum_1^n m_j \lambda_j,$$

and in the present case it is

$$2\ddot{\xi} = 2\frac{c}{\mu}\xi - \frac{1}{\mu}\sum m_j \lambda_j,$$

where μ is a constant Lagrange multiplier which can be determined with the aid of the isoperimetric constraint. The existence of a solution to this variational problem also remains an open question.

2. Particular Aspects of Such Isoperimetric Problems Illustrated by a Simple Example (n = 1)

The simultaneous Eqs. (1), the conditions (2) and (3), and the functional (4) become now

$$\dot{\eta} = b\eta + m\xi \qquad (m > 0), \tag{5}$$

$$\int_0^\infty \xi^2\, dt = \varkappa_1, \tag{6}$$

$$\eta(0) = \eta_0, \quad \eta(\infty) = 0, \tag{7}$$

$$I(\xi) = \int_0^\infty a\eta^2\, dt. \tag{8}$$

The Lagrange function is $F = a\eta^2 + \lambda\,\xi^2 + \lambda_1(\dot\eta - b\eta - m\xi)$. The quantities ξ and η satisfy the complete system of equations

$$\dot\eta = b\eta + m\xi, \quad \dot\xi = \frac{ma}{\lambda}\eta - b\xi, \tag{9}$$

whose characteristic equations have two roots,

$$\mu_{1,2} = \pm\sqrt{b^2 + \frac{m^2 a}{\lambda}}. \tag{9'}$$

Of course, whether the roots are real and different, or purely imaginary,or whether they are both zero depends on the values taken by λ. In view of the boundary conditions (7), it is obvious that we can only be interested in the first case, that is of real and different roots $\mu_1 \neq \mu_2$ ($\mu_2 < 0$), other words in the case of $b^2 + \frac{m^2 a}{\lambda} > 0$. Then

$$\eta = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t},$$

where c_1 and c_2, obtained from the boundary conditions (7),are $c_1 = 0$, $c_2 = \eta_0$.

Consequently

$$\eta = \eta_0 e^{\mu_2 t}, \quad \xi = \frac{\eta_0}{m}(\mu_2 - b)\,e^{\mu_2 t}, \tag{10}$$

$$\xi = \frac{\mu_2 - b}{m}\eta. \tag{10'}$$

In order to determine λ we make use of the isoperimetric constraint (6), by substituting instead of ξ, its value as given in (10). Thus, one obtains a quadratic equation for μ_2. By finding its root and subsequently comparing it with (9) we obtain for the determination of λ an algebraic equation

$$\mu_2 \equiv -\sqrt{b^2 + \frac{m^2 a}{\lambda}} = \left(b - \frac{\varkappa_1 m^2}{\eta_0^2}\right) - \sqrt{\frac{\varkappa_1 m^2}{\eta_0^2}\left(\frac{\varkappa_1 m^2}{\eta_0^2} - 2b\right)}. \tag{11}$$

Since the left-hand side is real, the expression under the square root $\frac{\varkappa_1 m^2}{\eta_0^2}\left(\frac{\varkappa_1 m^2}{\eta_0^2} - 2b\right)$ is, therefore, non-negative, that is

$$\frac{\varkappa_1 m^2}{\eta_0^2} - 2b \geqslant 0.$$

(It should be mentioned that it follows from (11') that $b - \frac{\varkappa_1 m^2}{\eta_0^2} < 0$, that is,the right-hand side of (11),is negative when $b > 0$. But when $b < 0$ then $b - \frac{\varkappa_1 m^2}{\eta_0^2}$ is always less than zero.) When $b < 0$ (the controlled object is stable) the above condition is always fulfilled, but when $b > 0$, only for

$$|\eta_0| \leqslant m\sqrt{\frac{\varkappa_1}{2b}} \equiv B_1. \tag{12}$$

Thus the isoperimetric constraint (6) causes the initial deviations of the phase coordinates to be subjected to the restriction (12) in the case of an unstable object; the restriction determines the domain of initial values such that the isoperimetric problem admits a solution. A similar analysis in the case of a system of the n-th order is very complex. From the physical aspect of the problem one can only presume that also in this case a constraint such as (12) should take place in the case of an unstable object. In such case the initial deviation of the system must be sufficiently small as otherwise the controller might "spend" the isoperimetric constant prior to the system

reaching the origin of the coordinate system; the boundary condition (3) would not be satisfied at the right end since without a controller the system would not return to zero. If there is such an initial deviation, the variational problem will have no solution. Another point of interest should be mentioned. As seen from (11), the root μ_2 of the characteristic equation, and consequently the characteristic of the regulator (see 10') depend on η_0. This was pointed out by a number of people [4, 5] and is also true in the general case as can easily be seen. Thus in the presence of the isoperimetric constraint (2), the control ξ depends on both η_i and η_{i0} ($i = 1,\ldots,n$). In this case it is necessary to have some additional information as regards the domain of initial values η_{i0} [4] when aiming at synthesis.

3. Additional Restriction

In addition to the isoperimetric constraint (6) the system is now also subjected to a restriction of the saturation type,

$$|\xi| \leqslant \bar{\xi}. \tag{13}$$

It shall be shown that the isoperimetric restriction is much stronger than the restriction (13) and causes the latter condition to be automatically fulfilled. In fact, the solution of the variational isoperimetric problem is $\xi = \sum_1^n p_i\eta_i$, where η_i are decreasing functions of time (the formula taken from [1], as previously mentioned, is valid here if c is replaced by λ). Consequently, if $|\xi(0)| = \bar{\xi}$ when $t = 0$, then for all $t > 0$, one has $|\xi(t)| \leq \bar{\xi}$.

The inequality $|\xi(0)| \leq \bar{\xi}$ is equivalent to

$$\left|\sum_1^n p_i \eta_{i0}\right| \leqslant \bar{\xi} \tag{14}$$

[$p_i = p_i(\eta_{j0})$, ($i, j = 1,\ldots,n$), as $p_i = p_i(\lambda)$, and λ in turn depends on the initial deviations of the phase coordinates], which means that the initial values of the phase coordinates have been subjected to a restriction.

We consider again our simple example, that is, the Eq. (5), the isoperimetric constraint (6), the boundary conditions (7), and the functional (8). We now add to it also the restriction (13).

By taking (10') into account, the inequality (14) now becomes

$$\left|\frac{\mu_2 - b}{m}\eta_0\right| \leqslant \bar{\xi}$$

or, on replacing μ_2 by its value from (11),

$$\left| -\frac{m\varkappa_1}{|\eta_0|} - \sqrt{\varkappa_1\left(\frac{m^2\varkappa_1}{\eta_0^2} - 2b\right)}\right| \leqslant \bar{\xi}, \tag{14'}$$

and hence

$$|\eta_0| \geqslant \frac{2\varkappa_1 m\bar{\xi}}{\bar{\xi}^2 + 2b\varkappa_1} \equiv A_1 > 0. \tag{15}$$

Thus when $b > 0$, η_0 must satisfy the inequalities (12) and (15) in view of the restrictions (6) and (13). The parameters of the Eq. (5) and the isoperimetric constant $\varkappa_1$ should be selected such that $B_1 \geq A_1$, as otherwise the inequality

$$A_1 \leqslant |\eta_0| \leqslant B_1$$

does not make sense. If $b < 0$ then $|\eta_0|$ need only satisfy the single inequality (15).

The phase trajectory is shown in Fig. 1a in the case of $b > 0$, the function $\xi = \xi(t)$ in the plane of (t, ξ) in Fig. 1b.

It follows from the inequality (15) that the initial deviations of the phase coordinates must be outside the interval $(-A_1, A_1)$ as the restriction (13) is valid. This actually means in practice that if $\eta_0 \in (-A_1, A_1)$, the controller — when the phase trajectory is given by the formula (10') — would not be fully utilized and also that $\int_0^\infty \xi^2 dt$ would be less than the given quantity κ_1. In order to prove it let $\eta_0 \in (-A_1, A_1)$. It follows from (10') that $\xi = k\eta$ where $k = \frac{\mu_2 - b}{m}$. For an $\eta_0 \in (-A_1, A_1)$ we have $k = k_1$ where $|k_1\eta_0| > \bar{\xi}$ as $\eta_0 \in (-A_1, A_1)$. But because of the restriction (13) the greatest possible value of $|\xi|$ is $\bar{\xi}$, and consequently $|\xi| = \bar{\xi}$; now a new angular coefficient $\tilde{k}$ of the phase trajectory should be selected with the aid of the condition $\bar{\xi} = |\tilde{k}|\,|\eta_0|$, i.e., $|\tilde{k}| = \frac{\bar{\xi}}{|\eta_0|}$. But obviously $|\tilde{k}| < |k_1|$, and consequently $|\tilde{\xi}| = |\tilde{k}|\,|\eta|$ is less than $|\xi_1| = |k_1|\,|\eta|$, i.e., and $\int_0^\infty \tilde{\xi}^2 dt < \int_0^\infty \xi_1^2 dt = \varkappa_1$.

It is obvious that for a weaker isoperimetric constraint

$$\int_0^\infty \xi^2 dt \leqslant \varkappa_1, \tag{17}$$

the inequality (15) would not be essential in the sense that for any $\eta_0 \in (-A_1, A_1)$, an angular coefficient of the phase trajectory can always be found such that the inequality (17) be fulfilled and also that $|\xi(0)| \le \bar{\xi}$.

It was proved that the optimal trajectory under both restrictions (6) and (13) cannot be of the form (10') if the inequality (15) is not satisfied at the initial instant. We shall now find the optimal trajectory for such a case, that is, when $|\eta_0| < A_1$. In view of what was previously said, it would be natural to make the assumption that $|\xi(t)| = \bar{\xi}$ $(0 \le t \le T)$ up to the time $t = T$ when another condition — now to be determined — begins to be valid.

When $t \in (0, T)$, the isoperimetric constraint can be written as

$$\int_t^\infty \xi^2 dt = \varkappa_1 - \bar{\xi}^2 t.$$

We introduce the notation $\kappa_1 - \overline{\xi^2}t = \kappa(t)$ and $\kappa(T) = \kappa_T$. Obviously T must satisfy the inequality

$$T \leqslant \frac{\varkappa_1}{\bar{\xi}^2}. \tag{17'}$$

The function $A(\kappa)$ is introduced by

$$A(\varkappa) = \frac{2m\bar{\xi}\varkappa}{\bar{\xi}^2 + 2b\varkappa};$$
$$\varkappa(0) = \varkappa_1, \quad A(\varkappa_1) \equiv A_1, \quad A(\varkappa_T) = A_T. \tag{18}$$

$A(\kappa)$ is a monotonically decreasing function of κ since

$$A'(\varkappa) = \frac{2m\bar{\xi}^3}{(\bar{\xi}^2 + 2b\varkappa)^2} > 0. \tag{18'}$$

The first time instant (if it does exist) is taken as the value of T and such that

$$|\eta(T)| = A_T \equiv \frac{2m\bar{\xi}(\varkappa_1 - \bar{\xi}^2 T)}{\bar{\xi}^2 + 2b(\varkappa_1 - \bar{\xi}^2 T)}. \tag{19}$$

If t = T were the initial time instant, then the condition analogous to (15) would be

$$|\eta(T)| \geqslant A_T, \tag{20}$$

consequently if the condition (19) is fulfilled, the optimal phase trajectory will, for all $t \geq T$, be the same as in (10') but starting from the point $\{\eta(T), \xi(T)\}$ where $|\xi(T)| = \bar{\xi}$, and also when $b > 0$ one must have

$$|\eta(T)| \leqslant B(T) \equiv m\sqrt{\frac{\varkappa_T}{2b}} = m\sqrt{\frac{\varkappa_1 - \bar{\xi}^2 T}{2b}} \tag{21}$$

[an inequality analogous to (12)].

The optimum phase trajectory in the case of $b > 0$ together with $|\eta_0| < A$, and also the curves $\xi = \xi(t)$ and $\eta = \eta(t)$ are shown in Fig. 2.

There still remains the problem concerning the actual existence of the time instant $t = T$, the latter being a root of the transcendental equation which in the case of $\xi = \bar{\xi}$ ($\eta_0 < 0$) is

$$\left|\left(\eta_0 + \frac{m}{b}\bar{\xi}\right)e^{bT} - \frac{m}{b}\bar{\xi}\right| = \frac{2m\bar{\xi}(\varkappa_1 - \bar{\xi}^2 T)}{\bar{\xi}^2 + 2b(\varkappa_1 - \bar{\xi}^2 T)}, \tag{22}$$

but in the case of $\xi = -\bar{\xi}$ ($\eta_0 > 0$) it becomes

$$\left|\left(\eta_0 - \frac{m}{b}\bar{\xi}\right)e^{bT} + \frac{m}{b}\bar{\xi}\right| = \frac{2m\bar{\xi}(\varkappa_1 - \bar{\xi}^2 T)}{\bar{\xi}^2 + 2b(\varkappa_1 - \bar{\xi}^2 T)}. \tag{22'}$$

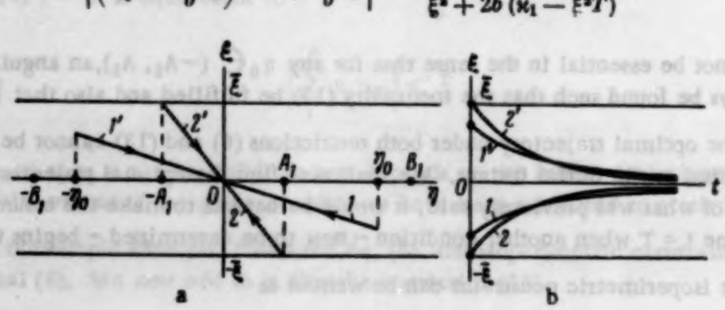

Fig. 1.

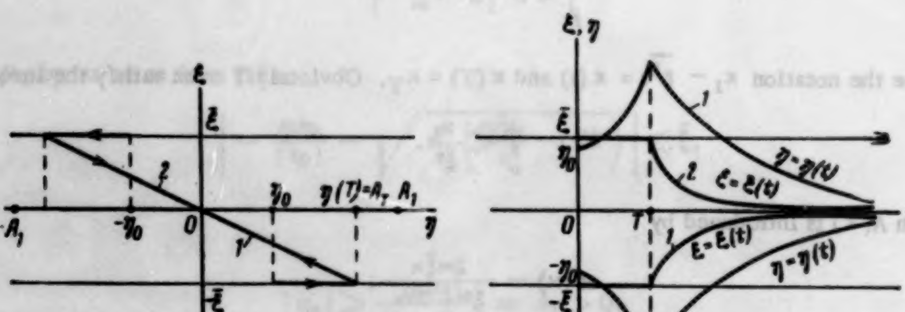

Fig. 2.

The equations (22) and (22') can be solved graphically. The abscissa T^* of the point of intersection (if any) thus obtained is the required instant T only if T^* either satisfies the inequality (17') (when $b < 0$), or (17') and (21) (when $b > 0$).

Thus the simultaneous imposition of the two constraints (2) and (13) leads to the following results.

SUMMARY

1. If the initial deviations of the system satisfy the inequality (14), the optimal phase trajectory is only determined by the restriction (2) because when (14) takes place, the trajectory is always within the boundary $|\varepsilon| = \bar{\varepsilon}$.

2. But if there is no constraint (2), the optimal trajectory also contains the boundary regions for which $|\varepsilon| = \varepsilon$, as shown in detail in our simple example.

Note: The optimal trajectory in this text is such that it only satisfies the necessary conditions of optimality though, strictly speaking, it would be pointless to use the formula without a prior investigation of the sufficient conditions.

In conclusion, I should like to express my deep gratitude to A. M. Letov for suggesting the problem as well as for his valuable advice.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**

REDUCTION OF A NONSTATIONARY LINEAR DIFFERENTIAL OPERATOR TO A SUM OF STATIONARY OPERATORS

A. N. Sklyarevich

(Riga)

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Original article submitted April 26, 1961

It is shown that an arbitrary nonstationary linear differential operator $\Phi(p, t)$ such that it can be represented as a finite power series with respect to the differentiation operator p, can be reduced to a sum of stationary operators with suitable kernels.

In [1] the question was posed how to reduce the nonstationary linear differential operator $\Phi(p, t)$ to a sum of stationary operators with suitable kernels. The problem shall be regarded as solved if the operators $\Phi_i(p)$ and the kernels $\varphi_i(t)$ are found such that for any function $x(t)$ the equality

$$\Phi(p, t)\, x(t) = \sum_i \Phi_i(p)\, [\varphi_i(t)\, x(t)] \tag{1}$$

will hold.

The reduction gives a relationship which would prove useful in obtaining an analogue of the system or in analyzing its structure, and also in examining the accuracy of the system's performance [2].

When i = 1, the study of a nonstationary system is reduced to that of a stationary one by replacing $y(t) = \varphi_1(t)x(t)$. By using the formula (1) one can develop a method of determining the stability of basically nonstationary systems (characterized by $i \geq 2$).

The procedure for reducing a nonstationary operator to the sum of stationary ones with suitable kernels was advanced in [1] in the case when the operator $\varphi(p, t)$ can be expressed in the form

$$\Phi(p, t) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} t^i p^j \qquad (c_{ij} = \text{const}),$$

that is, when it is a polynomial with respect to the differentiation operator p as well as with respect to the independent variable t.

It will be shown that these relations enable one to establish an algorithm of such a reduction in a more general case when the operator $\Phi(p, t)$ can be represented in the following form:

$$\Phi(p, t) = \sum_{j=0}^{n} \psi_j(t)\, p^j, \tag{3}$$

where $\psi_j(t)$ are known, differentiable functions.

As a preliminary step the condition shall be found under which the operator $\Phi(p, t)$ is reduced to a single operator $\Phi_1(p)$ with the kernel $\varphi_1(t)$. These conditions can be found if use is made of the relationship between the operators $\Phi(p, t)$ and $\Phi_1(p)$ and the function $\varphi_1(t)$ [1]:

$$\Phi(p, t) = \sum_{i=0}^{\infty} \frac{\varphi_1^{(i)}(t)\, \Phi_1^{(i)}(p)}{i!}, \tag{3}$$

according to which any nonstationary linear differential operator $\Phi(p, t)$ can be reduced to a single stationary operator if the functions $\varphi_1(t)$ and $\Phi_1(p)$ exist such that the relation (3) takes place.

When the formula (2) is valid it is reasonable to assume that the operator $\Phi_1(p)$ is a polynomial of degree $\underline{n}$ with respect to the differentiation operator $\underline{p}$:

$$\Phi_1(p) = \sum_{k=0}^{n} a_k p^k. \tag{4}$$

The quantities a_k do not depend on $\underline{t}$ as the operator $\Phi_1(p)$ is stationary.

The formula (4) implies

$$\Phi_1^{(i)}(p) = \sum_{k=i}^{n} \frac{k!}{(k-i)!} a_k p^{k-i} \qquad (i \leqslant n),$$

$$\Phi_1^{(i)}(p) = 0 \qquad (i > n),$$

and therefore the relation (3) is equivalent to

$$\Phi(p, t) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{(i+j)!}{i!\, j!} a_{i+j} \varphi_1^{(i)}(t)\, p^j, \tag{5}$$

where $j = k - i$.

By comparing the formulas (2) and (5) we obtain the equations

$$\psi_j(t) = \sum_{i=0}^{n-j} \frac{(i+j)!}{i!\, j!} a_{i+j} \varphi_1^{(i)}(t), \tag{6}$$

from which the kernel $\varphi_1(t)$ and the coefficients a_k of the operator $\Phi_1(p)$ can be determined, $\psi_j(t)$ being assumed known functions.

By putting $a_n = 1$ the generality is not reduced. Now the simultaneous equations (6) represent a closed system of $n + 1$ equations in $\underline{n}$ unknown quantities a_k($k = 0, 1, \ldots, n-1$) and one unknown function $\varphi_1(t)$.

It could be shown by mathematical induction that for any value of the subscript $\underline{k}$ the product $a_k\varphi_1(t)$ is given by the formula

$$a_k \varphi_1(t) = \sum_{i=0}^{n-k} \frac{(-1)^i (k+i)!}{k!\, i!} \psi_{k+i}^{(i)}(t). \tag{7}$$

One finds the relationship

$$\varphi_1(t) = \psi_n(t), \tag{8}$$

as a particular case of the formula (7) which one obtains when $k = n$. The formulas (7) and (8) determine completely the kernel $\varphi_1(t)$ and the operator $\Phi_1(p)$ in the case when the nonstationary operator $\Phi(p, t)$ is reducible to a single stationary operator.

In order to establish that such a reduction is possible we introduce the functions

$$r_k(t) = \sum_{i=0}^{n-k} \frac{(-1)^i (k+i)!}{k!\, i!} \psi_{k+i}^{(i)}(t). \tag{9}$$

In the special case of k = n we have

$$r_n(t) = \psi_n(t). \tag{10}$$

When the operator $\Phi(p, t)$ is reducible to a single stationary operator $\Phi_1(p)$ with kernel $\varphi_1(t)$, we obtain in accordance with the formulas (7) and (8)

$$r_k(t) = a_k \varphi_1(t) = a_k \psi_n(t), \tag{11}$$

where a_k = const.

It is obvious that all the functions $r_k(t)$ are linearly dependent on one another. Conversely, if all the functions $r_k(t)$ defined by the relations (9) are pairwise linearly dependent, then the nonstationary operator $\Phi(p, t)$ is reducible to the stationary one $\Phi_1(p)$ with kernel $\varphi_1(t) = r_n(t)$. The coefficients a_k of the operator $\Phi_1(p)$ can then be found from the formulas

$$a_k = \frac{r_k(t)}{r_n(t)}, \tag{12}$$

with the argument $\underline{t}$ being given any artibrary value.

The reduction of the operator $\Phi(p, t)$ to operator $\Phi_1(p)$ with kernel $\varphi_1(t)$ shall be denoted by

$$\Phi(p, t) = \Phi_1(p)\,[\varphi_1(t)], \tag{13}$$

where for the sake of brevity the arbitrary function x(t) as well as the right-hand part of the square bracket are ignored. In the general case, however, some of the $r_k(t)$ functions can be linearly independent and in this case the operator $\Phi(p, t)$ cannot be reduced to a single stationary operator $\Phi_1(p)$ whatever the kernel.

In this case it will prove useful to proceed as follows. We select the function $r_n(t)$ as kernel $\varphi_1(t)$ and choose arbitrarily in advance the quantities a_{1k} $(0 \leq k \leq n-1)$. We form the operator

$$F_2(p, t) = \Phi(p, t) - \Phi_1(p)\,[\varphi_1(t)], \tag{14}$$

where

$$\Phi_1(p) = \sum_{k=0}^{n} a_{1k} p^k \tag{15}$$

and $a_{1n} = 1$.

The operator $\Phi_1(p)\,[\varphi_1(t)]$ can be represented in the form

$$\Phi_1(p)\left[\varphi_1(t)\right] = \sum_{j=0}^{n} \psi_{1j}(t)\, p^j, \tag{16}$$

where the coefficients $\psi_{1j}(t)$ can be expressed in terms of the known functions $\Phi_1(p)$ and $\varphi_1(t)$ by means of the relations (6). In particular, when j = n we obtain $\psi_{1n}(t) = \varphi_1(t) = \psi_n(t)$ and consequently the highest degree terms with respect to the argument $\underline{p}$ of both operators $\Phi(p, t)$ and $\Phi_1(p)\,[\varphi_1(t)]$ are equal to one another. This implies that the operator $F_2(p, t)$ is of a lower order than the operator $\Phi(p, t)$.

By applying the same procedure with regard to the operator $F_2(p, t)$ we shall find the operator $F_3(p, t)$ defined by the formula

$$F_3(p, t) = F_2(p, t) - \Phi_2(p)\,[\varphi_2(t)], \tag{17}$$

this operator being of a lower order than the operator $F_2(p, t)$. Proceeding stepwise in this way, we arrive in the general case at an operator

$$F_l(p, t) = \varphi_l(t), \tag{18}$$

independent of the differentiation operator $\underline{p}$. The function $\varphi_l(t)$ can also be regarded as the operator $\Phi_l(p) = p^0$ with the kernel $\varphi_l(t)$. Adding the formulas (14), (17) and the subsequent ones up and including the formula (18), we obtain

$$\Phi(p, t) = \sum_{i=0}^{l} \Phi_i(p)\,[\varphi_i(t)], \tag{19}$$

which shows that the nonstationary operator $\Phi(p, t)$ can be reduced to a sum of stationary operators with appropriate kernels.

The reduction can be performed without evaluating the operators $F_2(p, t)$, $F_3(p, t)$,..., $F_l(p, t)$. The linearity of the formulas (9) with respect to the functions $\psi_{k+l}(t)$ implies that if $r_{2k}(t)$ and $r_k(t)$ are the functions characterizing the operators $F_2(p, t)$ and $\Phi(p, t)$ respectively, then taking into account the formulas (11), we obtain

$$r_{2k}(t) = r_k(t) - a_{1k} r_n(t). \tag{20}$$

In the case when the operator $\Phi(p, t)$ cannot be reduced to a single operator $\Phi_1(p)$ with kernel $\varphi_1(t)$, then some functions $r_{2k}(t)$ do not vanish identically. As in the first stage we take the function $r_{2k}(t)$ as the kernel $\varphi_2(t)$ such that the subscript $\underline{k}$ is the largest compatible with the function not vanishing identically. Specifying the values of a_{2k} with the first of the coefficients equal to unity, we determine the operator $\Phi_2(p)$, then the operator $\Phi_2(p)$ $[\varphi_2(t)]$, the functions $r_{3k}(t)$, etc., until all the functions $r_{l+1,k}(t)$ shall vanish simultaneously.

Thus the reduction of a nonstationary operator $\Phi(p,t)$ to a sum of stationary operators with corresponding kernels is performed on the functions $r_k(t)$ by relatively simple transformations. The number of successive stages in the transformations (that is, the integer l) is equal to the number of mutually independent functions $\psi_l(t)$ and such that any function $r_k(t)$ of the $r_k(t)$ system can be represented as a linear combination of the former functions.

One can see from this procedure that the quantities a_{ik} are not uniquely determined and that consequently there may be various (but equivalent) forms of representing a given nonstationary operator $\Phi(p, t)$ in the form of a sum of stationary operators. In particular, by putting $a_{ik} = 0$, we obtain $\Phi_i(p) = p^i$, $\varphi_i(t) = r_i(t)$, and therefore for an arbitrary function $x(t)$ one has

$$\Phi(p, t)\, x(t) = \sum_{i=0}^{n} p^i\,[r_i(t)\, x(t)]. \tag{21}$$

The representation (21), however, need not be the simplest one. When selecting the values of a_{ik} it is advisable to ensure the highest degree of simplification of the functions $r_{ik}(t)$.

Examples

1. Required to transform the operator

$$\Phi(p, t) = \cos t p^3 + (a_2 \cos t - 3 \sin t)\, p^2 + (a_1 \cos t - 2\, a_2 \sin t - 3 \cos t)\, p + a_0 \cos t - a_1 \sin t - a_2 \cos t + \sin t.$$

Now

$$\begin{aligned} \psi_3(t) &= \cos t, \\ \psi_2(t) &= a_2 \cos t - 3 \sin t, \\ \psi_1(t) &= a_1 \cos t - 2a_2 \sin t - 3 \cos t, \\ \psi_0(t) &= a_0 \cos t - a_1 \sin t - a_2 \cos t + \sin t. \end{aligned}$$

Therefore, by the formulas (9)

$$\begin{aligned} r_3(t) &= \psi_3(t) = \cos t, \\ r_2(t) &= \psi_2(t) - 3\,\psi_3'(t) = a_2 \cos t, \\ r_1(t) &= \psi_1(t) - 2\,\psi_2'(t) + 3\,\psi_3''(t) = a_1 \cos t, \\ r_0(t) &= \psi_0(t) - \psi_1'(t) + \psi_2''(t) - \psi_3'''(t) = a_0 \cos t. \end{aligned}$$

The functions $r_k(t)$ are all pairwise linearly dependent, and thus the operator $\Phi(p, t)$ is reducible to a single stationary

$$\Phi_1(p) = p^3 + a_2p^2 + a_1p + a_0.$$

with kernel $\varphi_1(t) = \cos t$.

2. Given the operator

$$\Phi(p, t) = tp^2 + (t^2 + a_1t + 2 + e^{\alpha t})\, p + ct^2 + (a_0 + 2)\, t + a_1 + e^{\alpha t}.$$

As one has in this case

$$\psi_2(t) = t,$$
$$\psi_1(t) = t^2 + a_1t + 2 + e^{\alpha t},$$
$$\psi_0(t) = ct^2 + (a_0 + 2)\, t + a_1 + e^{\alpha t},$$

therefore in accordance with the formulas (9)

$$r_2(t) = \psi_2(t) = t,$$
$$r_1(t) = \psi_1(t) - 2\psi_2'(t) = t^2 + a_1t + e^{\alpha t},$$
$$r_0(t) = \psi_0(t) - \psi_1'(t) + \psi_2''(t) = ct^2 + a_0t + (1 - \alpha)\, e^{\alpha t}.$$

The functions $r_2(t)$, $r_1(t)$ and $r_0(t)$ are pairwise linearly independent, and the operator $\Phi(p, t)$ therefore cannot be reduced to a single stationary operator $\Phi(p)$ with a appropriate base.

Following the previously described procedure, we put $\varphi_1(t) = r_2(t) = t$, $a_{11} = a_1$, $a_{10} = a_0$ and therefore

$$\Phi_1(p) = p^2 + a_1p + a_0.$$

To determine the functions $r_{2k}(t)$ we use the formulas (20):

$$r_{22}(t) = 0,$$
$$r_{21}(t) = t^2 + e^{\alpha t},$$
$$r_{20}(t) = ct^2 + (1 - \alpha)\, e^{\alpha t}.$$

Consequently

$$\varphi_2(t) = t^2 + e^{\alpha t},\ a_{20} = c,\ \Phi_2(p) = p + c.$$

Again applying the formulas (20) we obtain

$$r_{32}(t) = 0,\ r_{31}(t) = 0,\ r_{30}(t) = (1 - \alpha - c)\, e^{\alpha t}$$

and therefore

$$\varphi_3(t) = (1 - \alpha - c)\, e^{\alpha t},\ \Phi_3(p) = 1.$$

It has thus been established that for an arbitrary function $x(t)$ the following relation takes place:

$$\Phi(p, t)\, x(t) = (p^2 + a_1p + a_0)(tx) + (p + c)(t^2x) + p(e^{\alpha t}x) + (1 - \alpha)\, e^{\alpha t}x.$$

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SOME APPROXIMATE METHODS FOR SOLVING PROBLEMS OF OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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Approximate methods for solving problems of optimal control of distributed parameter systems are studied: the differential-difference methods, and also the method of moments.

I. In the synthesis of systems of optimal control of objects with distributed parameters there arises the complicated problem of determining the optimal control action [1, 2, 7]. Also, control action for such objects may be distributed in space and in time. Because of the complexity of the process taking place in distributed parameter systems and also because of insufficient development of control theory for such systems, it is sometimes difficult to obtain equations in a form sufficiently convenient for computation. In those cases in which we do not succeed in obtaining analytic expressions for the desired functions, along with direct methods of solution we can apply the method of reduction of partial differential equations to a system of ordinary differential equations and to finite difference equations (direct method and finite difference method, respectively), for which we already have a sufficiently well-developed theory of optimal processes [3-7].

We consider the problem of an optimal control system which is characterized in a broad sense by the extended one-dimensional equation of thermal conductivity

$$\frac{\partial Q}{\partial t} = a\,\frac{\partial^2 Q}{\partial x^2} \quad \left(a = \frac{\lambda}{c\rho}\right), \tag{1}$$

where $\underline{a}$ is the constant thermal conductivity coefficient, λ is the thermal conductivity coefficient, $\underline{c}$ is the thermal capacity coefficient, ρ is the specific weight.

The state of the system is described by the distribution function

$$Q = Q(x, t), \tag{2}$$

where $\underline{x}$ is the space coordinate ($0 \le x \le 1$), $\underline{t}$ is the time coordinate ($t_0 \le t \le t_1$).

The initial and boundary conditions are given by

$$Q(x, 0) = Q_0(x), \tag{3}$$

$$\left.\frac{\partial Q}{\partial x}\right|_{x=0} = \alpha\,[u(t) - Q(0, t)], \tag{4}$$

$$\left.\frac{\partial Q}{\partial x}\right|_{x=1} = 0. \tag{5}$$

Here $Q_0(x)$ is a known function, α is the constant heat transfer coefficient.

The problem of optimal control of this system can be formulated in various ways.

1 . Let there be given a certain function $Q^* = Q^*(x)$, defined in the same interval [0,1]. We are required to find such a control action $u = u(t)$ ($t_0 \le t \le t_1$), belonging to some defined class of functions with values in a region Ω, that at the end of the process at $t = t_1$ the deviation of the distribution function $Q(x, t_1)$ from the function $Q^*(x)$ will be minimal, i.e., that a certain functional

$$I = I[Q(x, t_1),\ Q^*(x)], \tag{6}$$

being a measure of this deviation, attains the smallest value.

In this regard the functional can be taken to be

$$I = \int_0^1 [Q^*(x) - Q(x, t_1)]^p\, dx \qquad (p > 0). \tag{7}$$

2 . Let there be given the same function $Q^* = Q^*(x)$. We are required to find such a control action $u = u(t)$ Ω that at the end of the process the condition

$$Q(x, t_1) = Q^*(x) \qquad \text{for} \quad 0 < x < 1 \tag{8}$$

is satisfied and the transition time of the process $T = t_1 - t_0$ is minimal.

However, physically it is clear that for a distributed parameter system requirement (8) is sometimes unrealizable. This may be interpreted thus: If the function $Q = Q(x, t)$ of argument $\underline{x}$ for fixed $t \in [t_0, t_1]$, is considered as the representative point of the system on the trajectory of the process in the functional phase space of the system (Hilbert space) with initial condition $Q(x, t_0) = Q_0(x)$, then it is necessary to choose such a control $u = u(t) \in \Omega$ that this phase trajectory in functional space hits the point $Q^*(x)$. In many cases if this exact coincidence is possible then the trajectory connecting points Q_0 and Q^* is unique and hence optimal.

For a more correct formulation of the problem it is not necessary to require an exact hit at the point Q^*, i.e., an exact satisfaction of equality (8); it is sufficient to restrict ourselves to hitting within a certain magnitude $\epsilon > 0$. More precisely this problem can be formulated in the following way.

3 . Let there be given the function $Q^* = Q^*(x)$. We are required to find such a control $u = u(t) \in \Omega$ ($t_0 \le t \le t_1$) that at the end of the process the condition

$$I_1 [Q(x, t_1),\ Q^*(x)] = \varepsilon$$

is satisfied, where $I_1[Q(x, t_1), Q^*(x)]$ is a given functional defining the measure of the deviation of the function $Q(x, t_1)$ from $Q^*(x)$, and that the transition time of the process will be minimal.

It is necessary to notice that the solution of the last problem does not exist for every $\epsilon > 0$. However, it is clear that in general this number $\epsilon > 0$ exists, although it must be sufficiently large.

The problem which has been considered is, for example, of practical interest in the determination of the best policy for heating metals in a continuous furnace and in a soaking pit.

By using the direct method we can reduce the problem under consideration to the general problem of optimal control studied in the papers indicated above. In fact, by dividing the interval $[0, 1]$ of the $\underline{x}$ axis into $\underline{n}$ equal parts by the points $x_0 = 0$, $x_1 = s, \ldots, x_n = 1$, where $s = 1/n$, and by replacing in Eq. (1) the second partial derivative with respect to $\underline{x}$ of the function $Q(x, t)$ by the second difference quotient, we obtain a differential-difference system in the functions $q_i(t)$ ($i = 0, 1, \ldots, n$):

$$\begin{aligned} \dot{q}_0 &= -(\sigma + \beta)\, q_0 + \sigma q_1 + \beta u, \\ \dot{q}_i &= \sigma (q_{i-1} - 2q_i + q_{i+1}) \qquad (i = 1, 2, \ldots, n-1), \\ \dot{q}_n &= \sigma (q_{n-1} - q_n) \end{aligned} \tag{9}$$

with initial conditions

$$q_i(t_0) = Q_0(is) \qquad (i = 0, 1, \ldots, n). \tag{10}$$

Here

$$\beta = \frac{\alpha}{c\rho s}, \qquad \sigma = \frac{a}{s^2}. \tag{11}$$

We can prove [10] that if we define u = u(t) then for sufficiently small s (sufficiently large n) the solution of the boundary value problem (1)-(5) is approximated with any degree of accuracy by the solution of the differential-difference system (9) with boundary conditions (10). Whatever be $\epsilon > 0$, an $s(\epsilon)$ can be found such that as soon as $s < s(\epsilon)$ we obtain

$$|Q(x_i, t) - q_i(t)| < \varepsilon \tag{12}$$

for every i = 0, 1,...,n simultaneously.

To obtain a finite system of n ordinary differential Eqs. (9) which include the control function u = u(t), it is necessary to state clearly the problem of optimal control corresponding to problems 1, 2, 3 formulated above.

For example, corresponding to problem 2 we must find such an admissible control action u = u(t) ($u \in \Omega$, $t_0 \le t \le t_1$) that the system passes from the initial state (10) to the given state

$$q_i(t_1) = Q^*(is) \qquad (i = 0, 1, \ldots, n) \tag{13}$$

in minimal time $T = t_1 - t_0$.

This problem can be solved with the help of the maximum principle [4]. It is easily proved that the optimal control in the case when the region of admissible values Ω is the interval $[-A, A]$ will have the form

$$u(t) = A \operatorname{sign} \psi_1(t), \tag{14}$$

where the functions $\psi_i(t)$ ($t_0 \le t \le t_1$) are defined by the homogeneous linear system

$$\begin{aligned} \dot\psi_0 &= (\sigma + \beta)\psi_0 - \sigma\psi_1 \\ \dot\psi_i &= \sigma(-\psi_{i-1} + 2\psi_i - \psi_{i+1}) \qquad (i = 1, 2, \ldots, n-1), \\ \dot\psi_n &= \sigma(-\psi_{n-1} + \psi_n). \end{aligned} \tag{15}$$

The initial values for functions $\psi_i(t)$ are determined from the condition of a hit on the representative point of system (9) at a given state, i.e., from condition (13).

In the case when the given state is $q_i(t_1) = 0$, i = 0, 1,...,n, it is proved in [4] that from any initial state $q_i(t_0)$ the system (9) can pass through the origin of coordinates optimally in the sense of minimal transition time for the process.

Thus we can consider the solution of the latter problem as being approximately the solution of problem 2. Here, of course, we must justify the convergence of this method and the existence of a solution of problem 2.

In some cases we are required to determine optimal processes in distributed parameter systems where the control action itself is distributed in space and has imposed on it constraints not only in time but also in the space variables. For example, sometimes it is essential to consider as inadmissible excessively large overshoots of the space variables for some physical quantities such as the temperature, pressure, electric field voltage, etc. In this case the approximate method of solution of the optimal control problem with the help of differential-difference equations can be useful since at present, evidently, there does not exist any other effective approach to solving this problem. As an example consider the heat transfer equation

$$b(x, t)\frac{\partial Q}{\partial t} + b(x, t)\,v(t)\frac{\partial Q}{\partial x} + Q = u(x, t) \tag{16}$$

between a stationary heating medium with temperature u(x, t) ($0 \le x \le l$, $t_0 \le t \le t_1$) where x is the space variable and t the time, and a material moving with velocity $v = v(t) \ge 0$ in the positive direction of the x axis which is heated while moving in the segment $0 \le x \le l$.

The heat state of this material is characterized by the temperature distribution function

$$Q = Q(x, t). \tag{17}$$

We will assume that the initial temperature distribution at time $t = t_0$ is given and is equal to

$$Q(x, t_0) = Q_0(x). \tag{18}$$

Moreover, the material enters the heating zone with a fixed temperature

$$Q(0, t) = 0 \text{ for } t_0 \leq t \leq t_1. \tag{19}$$

The optimal control problem for the heating zone consists of the determination of the function

$$u = u(x, t) \tag{20}$$

in the rectangle D ($0 \leq x \leq l$, $t_0 \leq t \leq t_1$) such that in spite of various perturbances in the heating process, caused by variations in the feeding rate of the material in the zone or by variations in the thermophysical parameter $b = b(x, t) > 0$ (for instance, variations in the thickness of the material layer, variations in thermal capacity, thermal conductivity, etc.), the deviation of the temperature of the material leaving the zone from some given technologically desired temperature will be minimal in some defined sense. It is necessary to minimize the functional

$$I = \int_{t_0}^{t_1} [Q^*(t) - Q(l, t)]^p \, dt \qquad (p > 0, \text{ even}) \tag{21}$$

where $Q^*(t)$ is a given function of time, and $v(t) \geq 0$, $b(x, t) > 0$ are assumed to be known for $t_0 \leq t \leq t_1$.

This problem has an importance significance in the heating of unfinished metals before passing them through a heating furnace.

Besides the constraint on the absolute value itself of the control i.e.,

$$A_1 \leqslant u(x, t) \leqslant A_2 \quad \text{for} \quad (x, t) \in D, \tag{22}$$

a constraint is imposed on the partial derivatives with respect to $\underline{x}$ of $u(x, t)$, i.e.,

$$B_1 \leqslant \frac{\partial u(x, t)}{\partial x} \leqslant B_2. \tag{23}$$

Here A_1, A_2, B_1 and B_2 are given constant numbers.

Physically this corresponds to the fact that, for example, in a continuous furnace it is impossible to create an excessively large overshoot in the temperature within the length of the furnace.

As can be seen, here we encounter a sufficiently difficult and unusual problem of optimal control of a distributed parameter system with constraints of type (22) and (23).

In order to reduce the partial differential equation to differential-difference equations we shall divide the interval $[0, l]$ of the $\underline{x}$ axis into $\underline{n}$ equal parts with the point $x_0 = 0$, $x_1 = s$, $x_2 = 2s, \ldots, x_n = l$, where $s = l/n$. In Eq. (18) we shall replace the partial derivatives with respect to $\underline{x}$ of the function $Q(x, t)$ by difference quotients, and we shall obtain a differential-difference system of $\underline{n}$ equations in the functions $q_i(t)$ in the form

$$b_i(t)\, \dot{q}_i + \frac{b_i(t)\, v(t)}{s} [q_i - q_{i-1}] + q_i = u_i(t) \qquad (i = 1, 2, \ldots, n). \tag{24}$$

From the constraint condition (19) we obtain

$$q_0(t) = 0 \text{ for } t_0 \leq t \leq t_1. \tag{25}$$

Equation (24) can be written in the form

$$\dot{q}_i = \beta q_{i-1} + \alpha_i q_i + u_i \qquad (i = 1, 2, \ldots, n). \tag{26}$$

In Eq. (26) we set:

$$\alpha_i = \alpha_i(t) = -\frac{1}{b_i(t)} - \frac{v(t)}{s}, \tag{27}$$

$$\beta = \beta(t) = \frac{v(t)}{s}, \tag{28}$$

$$b_i = b_i(t) = b(is, t),$$

$$u_i = u_i(t) = u(is, t) \qquad (i = 1, 2, \ldots, n). \tag{29}$$

Corresponding to (29) constraint (22) will have the form

$$A_1 \leqslant u_i(t) \leqslant A_2 \qquad (t_0 \leqslant t \leqslant t_1;\ i = 1, 2, \ldots, n), \tag{30}$$

and constraint (23) the form

$$B_1 \leqslant u_{i+1}(t) - u_i(t) \leqslant B_2 \qquad (t_0 \leqslant t \leqslant t_1;\ i = 1, 2, \ldots, n-1). \tag{31}$$

For this functional (21) must be replaced by the functional

$$I = \int_{t_0}^{t_1} [Q^*(t) - q_n(t)]^p \, dt. \tag{32}$$

Now for determining the optimal control action (29) under constraints (30) and (31) we must apply the maximum principle [3].

According to the maximum principle the function

$$H = H(q_i, \psi_i, u_i) = \psi_0 [Q^*(t) - q_n]^p + \sum_{i=1}^{n} \psi_i (\beta q_{i-1} + \alpha_i q_i + u_i) \tag{33}$$

must reach a maximum with respect to the arguments u_i ($i = 1, 2, \ldots, n$) for fixed values of the other arguments.

The functions $\psi_i = \psi_i(t)$, $i = 1, 2, \ldots, n$, must satisfy the linear homogeneous system of differential equations

$$\begin{aligned} \dot{\psi}_0 &= 0, \\ \dot{\psi}_i &= -\alpha_i \psi_i - \beta \psi_{i+1} \qquad (i = 1, 2, \ldots, n), \\ \dot{\psi}_n &= -p \psi_0 (Q^* - q_n)^{p-1} - \alpha_i \psi_n. \end{aligned} \tag{34}$$

It is evident that for a fixed ψ_i the maximum of function H will occur simultaneously with the reaching of a maximum of the function

$$R(\psi_i, u_i) = \sum_{i=1}^{n} \psi_i u_i. \tag{35}$$

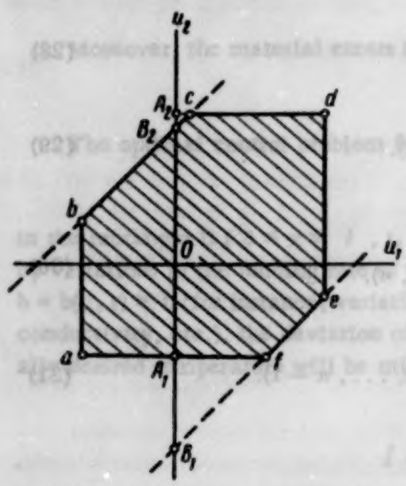
Thus we must find the maximum with respect to u_i of the linear form R with coefficients ψ_i under constraints (30) and (31).

Here the region Ω of admissible values of the functions u_i is a convex polyhedron in the n-dimensional Euclidean space $E(u_1, \ldots, u_n)$. As is known, this is a problem in linear programming. For example, if n = 2, constraints (30) and (31) will have the form

$$A_1 \leqslant u_1 \leqslant A_2, \quad A_1 \leqslant u_2 \leqslant A_2, \quad B_1 \leqslant u_2 - u_1 \leqslant B_2. \tag{36}$$

On the plane u_1, u_2, Eq. (36) is characterized by the polygon abcdef, representing the region. The maximum of the linear form R will, in this case, be reached at the vertices of this polygon, subject to the variations with time in the values of the coefficients ψ_1 and ψ_2.

II. The solutions of the linear equations of mathematical physics can be represented in integral form. The known methods of Fourier, Green, the method of integral transformations and successive approximations, etc., can be reduced to this form.



Thus the functions characterizing the state of the controlled object are connected with the control functions by integral relations.

For example, in the simplest one-dimensional case we have considered above, for problems 1, 2, 3, of optimal control of a heat transfer process, the state function Q(x, t) can be expressed in terms of the control action u(t), for zero initial conditions, in the following way:

$$Q(x, t) = \int_0^t K(x, t, \tau)\, u(\tau)\, d\tau, \tag{37}$$

where $K(x, t, \tau)$ is a known function, $t_0 = 0$, $t_1 = T$.

If at the moment t = T we are required to "hit" a given state $Q^*(x)$, then the problem of determining the control u = u(t) reduces to the solution of the integral equation of the first kind,

$$Q^*(x) = \int_0^T K(x, T, \tau)\, u(\tau)\, d\tau \qquad (0 \leqslant x \leqslant 1). \tag{38}$$

There is a theorem by Picard [11] which gives the necessary and sufficient conditions for the existence and uniqueness of the solution $u = u(\tau)$ $(0 \leq \tau \leq T)$ of Eq. (38). In fact, for fixed T let the function $K(x, T, \tau)$ be measurable and square-integrable in the rectangle $0 \leq x \leq 1$, $0 \leq \tau \leq T$, and let it be expanded in the series

$$K(x, T, \tau) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\, \psi_i(\tau), \tag{39}$$

converging to it in the mean, where

$$\lambda_i = \int_0^1\int_0^T K(x, T, \tau)\, \varphi_i(x)\, \psi_i(\tau)\, d\tau\, dx, \tag{40}$$

i.e.,

$$\int_0^1\int_0^T \Big[K(x, T, \tau) - \sum_{i=1}^{\infty} \lambda_i\varphi_i(x)\, \psi_i(\tau)\Big]^2 d\tau\, dx = 0. \tag{41}$$

The functions $\varphi_i(x)$ and $\psi_i(\tau)$ are measurable and square-integrable in the intervals $0 \leq x \leq 1$, $0 \leq \tau \leq T$, respectively, and form complete orthonormal systems.

$$\int_0^1 \varphi_i(x)\, \varphi_j(x)\, dx = \delta_{ij}, \qquad \int_0^T \psi_i(\tau)\, \psi_j(\tau)\, d\tau = \delta_{ij}. \tag{42}$$

Since the system of orthonormal functions $\varphi_i(x)$ $(i = 1, 2, \ldots;\ 0 \leq x \leq 1)$ is complete, the function $Q^*(x)$ $(0 \leq x \leq 1)$ can be expanded in terms of this system:

$$Q^*(x) = \sum_{i=1}^{\infty} \mu_i \varphi_i(x), \tag{43}$$

where

$$\mu_i = \int_0^1 Q^*(x)\, \varphi_i(x)\, dx. \tag{44}$$

We shall look for a solution of Eq. (38) in the form

$$u(\tau) = \sum_{i=1}^{\infty} \gamma_i \psi_i(\tau) \qquad (0 \leqslant \tau \leqslant T). \tag{45}$$

Substituting (43) and (45) into (38), by virtue of (42) we obtain

$$\sum_{i=1}^{\infty} \mu_i \varphi_i(x) = \sum_{i=1}^{\infty} \lambda_i \gamma_i \varphi_i(x). \tag{46}$$

Hence the desired coefficients γ_i in expansion (45) can be determined uniquely from

$$\gamma_i = \frac{\mu_i}{\lambda_i} \qquad (i = 1, 2, \ldots). \tag{47}$$

If the numerical series

$$\sum_{i=1}^{\infty} \gamma_i^2 = \sum_{i=1}^{\infty} \left(\frac{\mu_i}{\lambda_i} \right)^2 \tag{48}$$

converges, this signifies that there exists a unique solution (45) in the class of square-integrable functions. If series (48) does not converge, this solution does not exist. Note that if for any i, λ_i and μ_i are zero, then the term γ_i is also to be considered to equal zero. If for some i we have that $\lambda_i = 0$ and the corresponding $\mu_i \neq 0$, then, evidently, series (48) diverges and solution (45) does not exist. The uniqueness of the obtained solution follows from the completeness of the systems of functions $\varphi_i(x)$ and $\psi_i(\tau)$ $(i = 1,2,\ldots)$.

Having obtained solution (45) we shall be able to find out whether we can satisfy the necessary condition

$$u \in \Omega. \tag{49}$$

The smallest value T_{min} of all values of the upper limit T in Eq. (38), for which the unique solution (45) exists and satisfies condition (49), will be the smallest transition time of the process, and corresponding to this T_{min} function (45) will be the desired optimal control.

However, if there is no value of the upper limit T for which the integral Eq. (38) has a solution, then evidently the problem of a direct hit on the state $Q^*(x)$ has no meaning. This means that the "point" $Q^*(x)$ and the set of all states $Q(x, t)$ for various t, resulting from relation (37), lie in disjoint Hilbert subspace of functional spaces with a complete system of orthonormal functions $\varphi_i(x)$ $(i = 1,2,\ldots)$. Nevertheless, in this case, it is possible to consider the problem of determining such an optimal control

$$u = u(\tau), \qquad 0 \leqslant \tau \leqslant T, \qquad u \in \Omega, \tag{50}$$

that for a given fixed T it is possible to approach as "near" as possible to a given state $Q^*(x)$ in the sense of a metric space L_2, i.e., we must hit accurately the projection of the point $Q^*(x)$ on the subspace where the set of states $Q(x,t)$ is situated. This problem is analogous to the problem just studied.

We describe now an approximate method for solving the problem of an optimal "hit" on a given function $Q^*(x)$ or on its projection on some subspace of Hilbert space. The method requires that the kernel of Eq. (38) is approximated in the mean with a sufficient degree of accuracy by the degenerate kernel

$$K(x, T, \tau) \approx \sum_{i=1}^{n} \lambda_i g_i(x) h_i(\tau), \tag{51}$$

and for sufficiently large n, for an $\epsilon > 0$, satisfies the condition

$$\int_0^1 \int_0^T \left[K(x, T, \tau) - \sum_{i=1}^{n} \lambda_i g_i(x) h_i(\tau) \right]^2 d\tau\, dx < \varepsilon. \tag{52}$$

The question of convergence of the optimal approximate kernel in several variables, defined as the sum of products of functions every one of which depends on one variable, was studied in [12].

Expansion (51) is optimal if for a given $\epsilon > 0$ a smallest n can be obtained. The given number of n terms of the sum optimally approximates the kernel if the terms in the expansion are mutually conjugate functions of the unsymmetrical kernel $K(x, T, \tau)$, i.e., if $g_i(x)$ and $h_i(\tau)$ $(i = 1, 2, \ldots, n)$ are orthonormal systems in the first n eigenfunctions of the two integral equations with symmetric kernels,

$$\lambda g(x) = \int_0^1 M(x, \xi)\, g(\xi)\, d\xi, \quad \lambda h(\tau) = \int_0^T N(\tau, \theta)\, h(\theta)\, d\theta. \tag{53}$$

Here

$$M(x, \xi) = \int_0^T K(x, T, \tau)\, K(\xi, T, \tau)\, d\tau, \quad N(\tau, \theta) = \int_0^1 K(x, T, \tau) K(x, T, \theta)\, dx, \tag{54}$$

and λ_i $(i = 1, 2, \ldots, n)$ are eigenvalues which are the same for the two equations.

Having determined the eigenfunctions $g_i(x)$ and $h_i(\tau)$ $(i = 1, 2, \ldots, n)$ we can find the coefficients in the expansion of the function $Q^*(x)$ in terms of the functions $g_i(x)$ $(i = 1, 2, \ldots, n)$

$$\mu_i = \int_0^1 Q^*(x)\, g_i(x)\, dx. \tag{55}$$

Thus we can now raise the problem of hitting the function $\tilde{Q}(x)$ which is the projection of the function $Q^*(x)$ on the subspace corresponding to the orthonormal system of functions $g_i(x)$ $(i = 1, 2, \ldots, n)$, i.e.,

$$\tilde{Q}(x) = \sum_{i=1}^{n} \mu_i g_i(x). \tag{56}$$

If in Eq. (38) instead of $K(x, T, \tau)$ we use its approximation per formula (51), and if instead of $Q^*(x)$ we use $\tilde{Q}(x)$ per formula (56), then we get

$$\sum_{i=1}^{n} \mu_i g_i(x) = \sum_{i=1}^{n} \lambda_i g_i(x) \int_0^T h_i(\tau)\, u(\tau)\, d\tau. \tag{57}$$

Since the $g_i(x)$ $(i = 1, 2, \ldots, n)$ are linearly independent, to satisfy equality (57) it is necessary and sufficient to satisfy the equality

$$\nu_i = \int_0^T h_i(\tau)\, u(\tau)\, d\tau, \tag{58}$$

where

$$\nu_i = \frac{\mu_i}{\lambda_i} \quad (i = 1, 2, \ldots, n). \tag{59}$$

If we impose on the control $u = u(\tau)$ $(0 \le \tau \le T)$ the constraining condition

$$\|u\| \leqslant L \qquad (L < 0), \tag{60}$$

which states that the norm of function $u = u(\tau)$ should not exceed some positive number L, then the problem of determining the control $u = u(\tau)$ for which conditions (58), (59), (60) are satisfied, reduces to the so-called "L problem" studied in [8] and used earlier for the problem of optimal control of lumped parameter systems [9].

For this it can be assumed that the system of functions $h_i(\tau)$ $(i = 1, 2, \ldots, n;\ 0 \le \tau \le T)$ is not necessarily orthonormal; it is sufficient that they be linearly independent. This condition is useful when the expansion (51) of kernel $K(x, T, \tau)$ is not with respect to an orthonormal system but only with respect to a linearly independent system

$$g_i(x), \qquad h_i(\tau) \qquad (i = 1, 2, \ldots, n, \quad 0 \leqslant x \leqslant 1, \quad 1 \leqslant \tau \leqslant T).$$

Note that if the region Ω of admissible values of the control function $u(\tau)$ $(0 \le \tau \le T)$ is the closed interval $[-L, L]$, then this means that

$$\| u \| = \max_{0 \leqslant \tau \leqslant T} | u(\tau) | \leqslant L. \tag{61}$$

The results of [8] (page 171), applied to problems (58), (59), (60), show that the solution exists and is uniquely determinable from the condition of maximality with respect to u of the expression

$$u \sum_{i=0}^{n} a_i h_i(\tau) = \max \tag{62}$$

under the condition (61). Hence the desired optimal control

$$u = u(\tau) = L \operatorname{sign} \sum_{i=1}^{n} a_i h_i(\tau) \tag{63}$$

is uniquely determined.

The constant numbers a_i $(i = 1, 2, \ldots, n)$ are determined from the condition of minimality with respect to a_i of the expression

$$\int_0^T \left| \sum_{i=1}^{n} a_i h_i(\tau) \right| d\tau = \min \quad (i = 1, 2, \ldots, n) \tag{64}$$

under the condition

$$\sum_{i=}^{n} \nu_i a_i = 1. \tag{65}$$

We denote

$$f(T) = \left[\min_{a_i} \int_0^T \left| \sum_{i=1}^{n} a_i h_i(\tau) \right| d\tau \right]^{-1} \tag{66}$$

under the condition (65). Then the minimal transition time for the process is determinable as the smallest root of the equation

$$f(T) = L. \tag{67}$$

In conclusion we note that the reduction procedure is valid also in the case of some control actions when the action depends not only on time but also on the space coordinates in the presence of nonzero initial conditions. In this case the problem reduces to the problem of moments in multidimensional space [13] and to the notion of kernels in many variables in the form of a sum of products of functions each one of which depends on a single variable [12].

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DYNAMICS OF RELAY SELF-OSCILLATING EXTREMUM CONTROL SYSTEMS

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The present article is concerned with the dynamics of relay self-oscillating extremum control systems in the presence of random disturbances. The approximation method was used for solving this problem. Approximate expressions for estimating the probability of losing the extremum are given.

Self-oscillating extremum control systems are technically the simplest systems of this type; however, some of their dynamic properties still remain unexplained, which does not allow us to determine in a sufficiently positive manner the possible scope of practical application of such systems. It is intuitively clear that the presence of intensive random noise can greatly distort the pattern of the control system's operation and even cause a total loss of the possibility of stable extremum tracking. Therefore, a consideration of the noise effect should be obligatory in a preliminary analysis of the dynamic properties of such systems when they are designed.

It is shown in the present paper that, under certain assumptions, such an analysis can be performed without resorting to special investigations and using only data obtained in the dynamic calculations which are usually performed for determining the frequency and amplitude of self-oscillations in the absence of noise.

A relay extremum control system, the block diagram of which is shown in Fig. 1a, is considered below. The system's motion is described by the equations

$$x = \frac{1}{p} K_1(p)\,\eta, \qquad y = f(x + \zeta_1), \tag{1}$$
$$\xi = pK_2(p)\,y, \qquad \eta = F(\xi + \zeta_2).$$

Here, ζ_1 and ζ_2 are external disturbances, which are respectively reduced to the input of the system to be controlled [with respect to the static characteristic $f(x)$] and to the input of the nonlinear control device (in the form of a series relay-to-trigger coupling) with the ambiguous characteristic $F(\xi)$ (Fig. 1b), which was described in [1].

The $K_1(p)$ and $K_2(p)$ operators characterize the dynamic properties of the system to be controlled with respect to its input and output. It is assumed that, near the extremum, the static characteristic of the system to be controlled can be approximated by a quadratic parabola:

$$f(x) = kx^2.$$

The determination of the steady-state parameters of the described system for $\zeta_1, \zeta_2 = 0$ was performed in [1] by using the describing-function approach and, later, in [2], by using the approximation method. In both methods, the procedure is reduced to the determination of the dependences of the basic steady-state parameters, namely, the frequency ω (or the half-period T) of self-oscillations and of the hunting amplitude A, on the circuit parameters, in particular, the actuation level $\varkappa_0$ of the relay:

$$\omega = \omega(\varkappa_0), \qquad T = T(\varkappa_0), \qquad A = A(\varkappa_0). \tag{2}$$

These dependences are rather smooth, and they can be linearly approximated without difficulties in a small range of $\varkappa_0$ variations.

I. Effect of Noise at the Output of the System to be Controlled

We shall consider the case where noise acts only at the output of the system to be controlled, when, consequently, it can be reduced to the input of the control device by linear transformation. The effect of noise ζ_2 on the system's operating conditions can manifest itself only in a change of the actual operating level of the control device's relay, i.e., the relay will in this case be actuated not under the conditions $\xi = \kappa_0$, $p\xi > 0$, but under conditions where

$$\xi + \zeta_2 = \varkappa_0, \qquad p(\xi + \zeta_2) > 0,$$

or

$$\xi = \varkappa_0 - \zeta_2 = \varkappa, \qquad p\xi > p\zeta_2.$$

Assume that the noise level is rather low:

$$\zeta_2 \ll A. \tag{3}$$

Then, the changes in the operating conditions will be reduced to small fluctuations of the moments of switching and to small amplitude fluctuations, i.e., the limiting cycle will be reduced to a rather narrow ring. It is obvious that an exact investigation of the system's dynamics under these conditions (see, for instance [3, 4]) can be reduced to finding the steady-state solution of a system of equations in terms of finite differences.

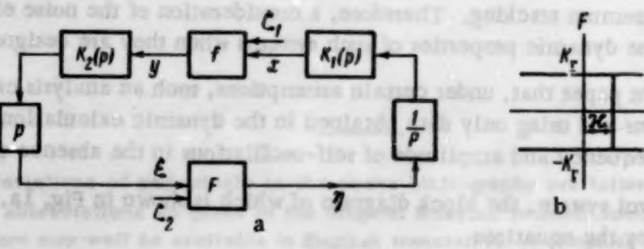

Fig. 1.

For complex systems, this procedure is rather cumbersome, although the basic difficulties can be overcome. A simpler approach can be used in two cases.

1. We shall assume that the variation of noise ζ_2 and the corresponding changes in the fluctuating level κ take place rather slowly; more accurately, we shall consider that the κ values for both switchings that occur during a single period coincide. Then, by using the method described in [5], it can be shown that the basic quantities that characterize the periodic component of motion will obey the following laws:

$$\omega^* = \omega(\varkappa), \qquad T^* = T(\varkappa), \qquad A^* = A(\varkappa), \tag{4}$$

where ω^*, T^*, and A^* are slowly-changing random functions of time (or, more accurately, random sequences).

On the basis of (3), we have

$$\Delta\omega = \omega^* - \omega = -\left(\frac{\partial\omega}{\partial\varkappa}\right)_{\varkappa=\varkappa_0}\zeta_2,$$
$$\Delta T = T^* - T = -\left(\frac{\partial T}{\partial\varkappa}\right)_{\varkappa=\varkappa_0}\zeta_2,$$
$$\Delta A = A^* - A = -\left(\frac{\partial A}{\partial\varkappa}\right)_{\varkappa=\varkappa_0}\zeta_2.$$

The root-mean-square values of the fluctuations $\Delta\omega$, ΔT, and ΔA are related to the root-mean-square noise values σ_2 by the following elementary relationships:

$$\sigma_\omega^2 = \left(\frac{\partial\omega}{\partial\varkappa}\right)^2_{\varkappa=\varkappa_0}\sigma_2^2, \quad \sigma_T^2 = \left(\frac{\partial T}{\partial\varkappa}\right)^2_{\varkappa=\varkappa_0}\sigma_2^2, \quad \sigma_A^2 = \left(\frac{\partial A}{\partial\varkappa}\right)^2_{\varkappa=\varkappa_0}\sigma_2^2. \tag{5}$$

The values of the partial derivatives that enter (5) can be found graphically or analytically. For instance, let us consider the simplest case, where

$$K_1(p) = 1, \quad K_2(p) = \frac{1}{T_2 p + 1}. \tag{6}$$

The equation for determining the periods can be found by using the approximation method (2); it has the following form:

$$\frac{\varkappa}{2kk_r^2} = \frac{T}{1 - e^{-\frac{T}{T_2}}} - T_2 - \frac{T}{2}, \tag{7}$$

whence

$$\frac{\partial T}{\partial \varkappa} = \frac{1}{kk_r^2}\,\frac{\left(1 - e^{-\frac{T}{T_2}}\right)^2}{1 - \frac{2T}{T_2} e^{-\frac{T}{T_2}} - e^{-\frac{2T}{T_2}}}, \tag{8}$$

$$\frac{\partial A}{\partial \varkappa} = \frac{1}{2} k_r \frac{\partial T}{\partial \varkappa}, \qquad \frac{\partial \omega}{\partial \varkappa} = -\frac{2\pi}{T^2}\frac{\partial T}{\partial \varkappa}.$$

Here, k_r is the signal value at the relay's output (Fig. 1b).

2. For other, slightly different, assumptions,* this problem can be solved in the same elementary way.

We shall consider only systems of this type, where:

a) The times of the transient processes in the linear parts of the system are considerably shorter than the half-period T, i.e.,

$$\lambda_i T \gg 1, \tag{9}$$

where λ_i are the roots of the denominators $K_1(p)$ and $K_2(p)$;

b) The successive noise values at the moments of level intersection are independent random quantities.

It is physically clear that, under these assumptions, the excitation of each half-cycle can be considered separately. The values of T^*, ω^* and A^* for the successive half-cycles form a sequence of independent random quantities, each of which is related to the random actuation level by the expressions (4).

Thus, relationships (5)** can be used also in this case for determining the root-mean-square deviations σ_T, σ_A, and σ_ω; however, in determining the partial derivatives by using equations of the type (8), it is advisable to introduce simplifications right away by taking into account the initial assumption (9). In particular, for a system with operators of the type (6), we simply have

$$\frac{\partial T}{\partial \varkappa} = \frac{1}{kk_r^2}, \quad \frac{\partial A}{\partial \varkappa} = \frac{1}{2kk_r}, \quad \frac{\partial \omega}{\partial \varkappa} = -\frac{2\pi}{T^2}\frac{1}{kk_r^2}. \tag{10}$$

II. Dynamic Action of Noise which can be Linearly Reduced to the Input of the System to be Controlled

We shall now consider the more complicated problem where noise acts at the input of the system to be controlled ($\zeta_1 \neq 0$, $\zeta_2 = 0$).

This problem can also be approximately investigated on the basis of the two schematization variants which were used above in analyzing the dynamics of a system where noise acts only at the output of the system to be controlled.

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* The problem was also solved in [6] under the same assumptions, but using a much more cumbersome procedure.

** Although this result has a semiheuristic character, it can be substantiated within the framework of the above-mentioned exact theory.

A. We shall assume that $\zeta_1(t)$ can be approximated by a sectionally-linear process (by a triangular function of time) with slope values changing in random fashion for different periods; in other words, we shall consider the drift rate of the characteristic to be a random quantity that assumes a constant value during each period (Fig. 2).

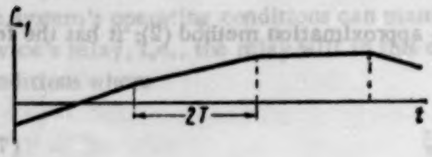

Fig. 2.

In this, the variation of the drift rate is assumed to be slow. Then, for determining the fluctuations of the self-oscillation parameters, we can use the results obtained in calculating the system's operating conditions for a strictly constant drift rate. In this case, naturally, the oscillation symmetry is lost, but the calculations can be performed by using, for instance, the approximation method in a way similar to that used in [2].

The equation of periods for the simplest system with operators of the type (6) has the following form:

$$F(\varkappa, \lambda, T') = \frac{\varkappa}{2kT_2(k_r^2 - \lambda^2)} - \frac{k_r + \lambda}{4k_r T_2} T'\left(\mathrm{cth}\,\frac{T'}{2T_2} + \mathrm{cth}\,\frac{T''}{2T_2}\right) + 1 = 0, \tag{11}$$

where

$$\lambda = \frac{\partial \zeta_1}{dt}, \quad T'' = T' \frac{k_r + \lambda}{k_r - \lambda},$$

and T' and T" are the time interval values corresponding to the final element's motion in one direction.

Considering that $\lambda = \lambda_0 + \Delta\lambda$, where $\Delta\lambda$ is a small random quantity whose values are independent for each period, we obtain

$$\frac{\sigma_{T'}}{\sigma_\lambda} = \left| \frac{\partial F}{\partial \lambda} \left(\frac{\partial F}{\partial T'} \right)^{-1} \right|_{\lambda=\lambda_0,\ T'=T'_0}. \tag{12}$$

Here, T'_0 is the solution of the equation $F(\lambda_0, T') = 0$.

The partial derivatives have the following values:

$$\left.\frac{\partial F}{\partial \lambda}\right|_{\substack{\lambda=\lambda_0,\\ T'=T'_0}} = \frac{\varkappa\lambda_0}{kT_2(k_r^2 - \lambda_0^2)} + \frac{(T'_0)^2}{4T_2^2(k_r - \lambda_0)\,\mathrm{sh}\,\frac{T'_0}{2T_2}} + \frac{1}{k_r + \lambda_0}\left[1 + \frac{\varkappa}{2kT_2(k_r^2 - \lambda_0^2)}\right],$$

$$\left.\frac{\partial F}{\partial T'}\right|_{\substack{\lambda=\lambda_0,\\ T'=T'_0}} = \frac{1}{T'_0}\left[1 + \frac{\varkappa}{2kT_2(k_r^2 - \lambda_0^2)}\right] + \frac{T'_0}{8k_r T_2^2}\left(\frac{k_r - \lambda_0}{\mathrm{sh}\,\frac{T'_0}{2T_2}} + \frac{k_r + \lambda_0}{\mathrm{sh}\,\frac{T''_0}{2T_2}}\right).$$

For the second half-period, we have

$$\sigma_{T''} = \sigma_{T'} \frac{k_r + \lambda_0}{k_r - \lambda_0}.$$

We shall also find the expression for the root-mean-square values of fluctuations of the maximum coordinate deflection at the input of the system to be controlled. From the solution of the determinate problem (calculation in the presence of drift with a constant rate λ), we find

$$A = (k_r + \lambda)\left[-\frac{T' e^{-\frac{T'}{T_2}}}{1 - e^{-\frac{T'}{T_2}}} + T_2 + \frac{\varkappa}{2k(k_r + \lambda)^2}\right], \tag{13}$$

where A is the hunting amplitude as before.

The relationship between fluctuations A and T* is given by the equation

$$\frac{\sigma_A}{\sigma_{T'}} = N(T_0'), \tag{14}$$

where

$$N(T_0') = (k_r + \lambda_0)\frac{e^{-\frac{T_0'}{T_2}}}{1 - e^{-\frac{T_0}{T_2}}}\left|1 - \frac{\frac{T_0'}{T_2}}{1 - e^{-\frac{T_0}{T_2}}}\right|.$$

B. Let us consider a system where the conditions (9) are satisfied. In this case, the following signal arises at the output of the system to be controlled:

$$y = k(x + \zeta_1)^2 \tag{15}$$

or, if we neglect small quantities of the second order,

$$y = kx^2 + 2kx\zeta_1. \tag{16}$$

Let us denote

$$z = 2kx\zeta_1. \tag{17}$$

The values of noise which is reduced to the input of the control device (at the moment of switching T*) is related to $\underline{z}$ by the following expression (with an accuracy to small quantities of the first order):

$$\zeta_{2re}(T^*) = \int_0^\infty W_2(\tau)\, z(T - \tau)\, d\tau, \tag{18}$$

where $W_2(\tau)$ is a pulse function which corresponds to the $pK_2(p)$ operator.

On the basis of the accepted assumption (9), we have

$$\zeta_{2re}(T) \approx \int_0^T W_2(\tau)\, z(T - \tau)\, d\tau = \int_0^T W_2(T - \tau)\, z(\tau)\, d\tau, \tag{19}$$

i.e., in calculating $\underline{z}$, only the $x_1(t)$ values for the given half-period can be used.

The root-mean-square value of ζ_{2re} can be determined from the equation

$$\sigma_2^2 = 4k^2\sigma_1^2 \int_0^T\int_0^T W_2(T - \tau_1)\, W_2(T - \tau_2)\, x(\tau_1)\, x(\tau_2)\, \varphi(\tau_1 - \tau_2)\, d\tau_1 d\tau_2, \tag{20}$$

where $\sigma_1^2\varphi_1(\tau)$ is the autocorrelation function of the $\zeta_1(t)$ process.

For a system with operators which are given by (6) and $\varphi_1(\tau) = e^{-\alpha|\tau|}$, the calculations can readily be performed in explicit form by expressing σ_2 in terms of the system's parameters and the exponent α; the equation thus obtained is rather unwieldy: however, for the case where $\alpha T_2 \gg 1$, we obtain,

$$\sigma_2 \approx \sigma_1 \frac{Tkk_r}{T_2}. \tag{21}$$

After calculating σ_2 by using (20), the root-mean-square values of the fluctuations of the oscillating regime's parameters are calculated by using the same equations (5) that were used for noise linearly reducible to the input of the control device.

It is obvious that the described calculation method can readily be extended to the case where a constant-rate drift of the characteristic of the system to be controlled is present besides random disturbances.

III. Estimate of the Extremum Loss Probability

Due to the structural characteristics of the extremum control system under consideration, noise action can cause irreversible errors, i.e., it can cause a drift of the system away from the extremum.

We shall consider a control device which consits of the following series-connected elements: a two-position relay, a differentiating network which produces pulses for every switching of the relay, and a trigger that is actuated by pulses with the same polarity. A signal $\xi(t)$ acts at the input of the control device; after the signal reaches the level κ, the relay is actuated (Fig. 3), the pulse at the differentiating element's input causes the tip-over of the trigger, and the system's motion is reversed.

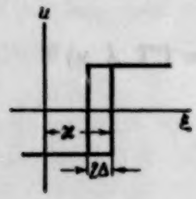

Fig. 3.

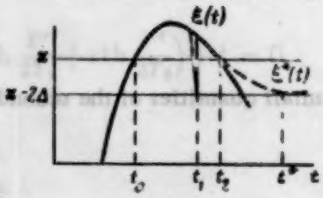

Fig. 4.

For the next trigger actuation, it is necessary that the signal $\xi(t)$ be reduced to the $\kappa-2\Delta$ level, i.e., that the relay be returned to its initial position and that the possibility of a new triggering pulse arriving at the trigger's input be provided. In the case of a minimum (maximum) extremum, the instants of time when the κ level is reached from below (or from above) must alternate with the moments when the $\kappa-2\Delta$ values are reached from above (from below).

The error of the system arises in the following manner. At the instant of time t_0, the control device operates, but, until the moment t_2, the signal at the control device's input still exceeds the level κ due to inertia. If a pulse signal whose intensity is sufficient for crossing the $\kappa-2\Delta$ level acts during the time interval (t_0, t_2), the relay of the control device will first reestablish the initial state, after which it will cause the trigger to operate and the system to reverse its motion.

Consider the instant of time t_1 which is sufficiently removed from t_0 for the curve $\xi(t)$ only to touch the $\kappa-2\Delta$ level without crossing it (Fig. 4) when the control device is actuated at this instant (t_1).

If a noise pulse arrives during the time interval (t_0, t_1), where it is assumed that $t_1 \leq t_2$, the system commits a basic error and drifts away from the extremum.

An approximate estimate of the probability of the system's drift can be obtained for the following assumptions concerning the character of noise:

a) The error that is reduced to the input of the control device has a pulse character, and its correlation time is very small in comparison with the self-oscillation period;

b) The disturbance has a Gaussian distribution with a small root-mean-square value.

Under the above assumptions with respect to the character of noise, it is possible to use certain results which give the average number of times a steady-state disturbance with a Gaussian probability density crosses the constant level.

If there is a steady-state random function $\zeta_2(t)$ with a Gaussian probability density

$$f(\zeta_2) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{\zeta_2^2}{2\sigma_2^2}},$$

the average number of crossings of the assigned constant level A per unit time is determined by the expression (see, for instance [7])

$$N_{av} = \frac{\omega_{II}}{2\pi} e^{-\frac{A^2}{2\sigma_2^2}}, \tag{22}$$

where $\omega_{II}^2 = -\varphi_2''(0)$, while $\varphi_2(\tau)$ is the correlation factor of the steady-state process $\zeta_2(t)$.

Under steady-state conditions and in the absence of external noise, the signal $\xi(t)$ acts at the input of the control device.

We can write the expression

$$N(\eta) = \frac{\omega_{II}}{2\pi} \exp\left\{-\frac{[2\Delta + \xi(\eta)]^2}{2\sigma_2^2}\right\}, \tag{23}$$

which gives the average number of times the steady-state disturbance $\zeta_2(t)$ crosses the assigned level, which is determined by the fixed instant of time η.

The arrival of pulses at any instant of time within the (t_0, t_1) interval is equally probable, and, consequently, the average number of crossings of the $\varkappa$-2Δ level per unit time is determined by the expression

$$N_{av} = \frac{\omega_{II}}{2\pi} \frac{1}{T} \int_{t_0}^{t_1} \exp\left\{-\frac{[2\Delta + \xi(\eta)]^2}{2\sigma_2^2}\right\} d\eta, \tag{24}$$

where T is the self-oscillation half-period.

The $1/N_{av}$ values determine the average operating time of the system until its first error and drift away from the extremum occur. The system's parameters enter the expression for $1/N_{av}$ through the T, $\xi(t)$, t_1, t_2 values.

As an example, we shall consider a very simple system with operators of the type (6). Under steady-state conditions, we have $(t_0 = 0)$

$$x(t) = k_r\left(t - \frac{T}{2}\right),$$
$$\xi(t) = \left[\varkappa + 2kk_r^2\left(T_2 + \frac{T}{2}\right)\right]e^{-\frac{t}{T_2}} + 2kk_r^2\left(t - T_2 - \frac{T}{2}\right). \tag{25}$$

Assume that an external pulse caused the control device's relay to operate at the instant of time t_1. Then, for disturbed motion at the input of the control device, we have

$$x^* = k_r\left(t_1 - \frac{T}{2} - t\right), \qquad \xi^* = De^{-\frac{t}{T_2}} + 2kk_r^2\left(t - T_2 + \frac{T}{2}\right), \tag{26}$$

where the time is counted from the moment t_1. We shall determine the value of t_1, the instant in time t^* when the curve ξ touches the $\varkappa$-2Δ level, and the constant D from the conditions

$$\xi^*(0) = \xi(t_1), \quad \xi^*(t^*) = \varkappa - 2\Delta, \quad \frac{d\xi^*(t^*)}{dt} = 0. \tag{27}$$

As a result, we find $D = 2kk_r^2 T_2 e^{\frac{t^*}{T_2}}$ and the equations for determining t_1 and t^*:

$$e^{\frac{t^*}{T_2}} = \left(\frac{\varkappa}{2kk_r^2} + T_2 + \frac{T}{2}\right)e^{-\frac{t_1}{T_2}} + 2t_1 - T, \qquad t^* = \frac{\varkappa - 2\Delta}{2kk_r^2} + t_1 - \frac{T}{2}. \tag{28}$$

Naturally, N_{av} can be calculated only approximately by solving the system (28) numerically and by integrating (24), where $\xi(t)$ is determined by using Eq. (25).

IV. Certain Results of the Experimental Investigation of the System

The basic relationships between the root-mean-square values of the self-oscillation frequency fluctuations were checked experimentally. The work was performed with an ÉMU-6 electronic apparatus with a specially constructed device, which was described in section III. A shot noise generator was used as the source of random disturbances [8].

TABLE 1

k_p, v	T_2, sec	ω, 1/sec	σ_2, v	σ_ω/σ_2, (1/v)·sec experiment	σ_ω/σ_2, (1/v)·sec calculation
10	4.0	1.1	0.68	0.068	0.063
			8.3	0.026	
10	1.0	2.1	1.02	0.194	0.233
			2.54	0.154	
			8.3	0.120	
15	1.0	3.3	1.02	0.153	0.258
			2.54	0.185	
			8.3	0.152	

TABLE 2

k_p, v	ω, 1/sec	σ_2, v	σ_ω/σ_2, (1/v)·sec experiment	σ_ω/σ_2, (1/v)·sec calculation
10	1.1	1.41	0.024	0.027
		8.3	0.013	
15	1.8	1.41	0.051	0.030
		8.3	0.018	
24	3.0	1.41	0.069	0.031

The noise correlation factor was approximated by the exponential function

$$\varphi(\tau) = e^{-4.2|\tau|}.$$

For low random noise levels, the results of theoretical calculations were in satisfactory agreement with experimental data.

For random noise at the control device's input, the calculations were performed by using Eq. (10). The calculation results and their comparison with experimental data are given in Table 1, where it is assumed that $k = 0.03$ $(1/v)\cdot$sec and $\kappa = 3.6$ v.

For random disturbances at the input of the system to be controlled, the calculations were performed by using Eq. (21). A comparison between the experimental and theoretical data is given in Table 2, where it is assumed that $T_2 = 4$ sec, $k = 0.03$ $(1/v)\cdot$sec and $\kappa = 3.6$ v.

Comparisons show that there is satisfactory agreement between the calculation results and experimental data for low-level random disturbances.

The results of theoretical calculations become more accurate as ω decreases (T increases), i.e., as the fulfillment of conditions (9) becomes more perfect. It should be noted that the T/T_2 ratio only slightly exceeds or even becomes less than unity for all the parameters considered above.

The acceptable accuracy of the obtained calculation estimates indicates that the method used is not o[illegible] critical with respect to the fulfillment of the damping conditions (9).

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**

COMPLEX PERIODIC OPERATING CONDITIONS IN RELAY EXTREMUM SYSTEMS

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The present article is concerned with the problem of determining the conditions and regions of existence of complex periodic operating conditions in extremum systems for two methods of approximating the characteristic of the nonlinear part of the system to be controlled. The problem of investigating the stability of complex periodic operating conditions is considered. Recommendations concerning the practical application of complex periodic operating conditions are given.

1. Introduction

Relay extremum control systems where the absolute value of the control element's speed is constant while its sign is reversed after every passage through the extremum in correspondence with the assigned switching law constitute one of the types of extremum control systems with independent scanning. The block diagram of such a system is shown in Fig. 1a. The system to be controlled consists of a linear unit, LU, with the transfer function $W_0(p)$ and a nonlinear unit, NU, with the characteristic $z = F(y)$, the only known fact concerning it is that it has an extremum. The control device of the extremum regulator ER has the characteristic shown in Fig. 1b. As was shown in [1], it can be represented by a series connection of an ordinary relay with hysteresis and a ratchet mechanism. An ordinary servomotor with the transfer function $W_{fe}(p) = k_1/p$ serves as the final element FE in the system.

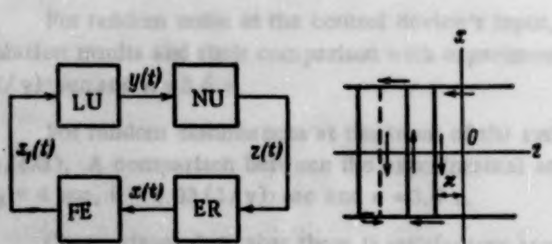

Fig. 1.

Simple as well as complex periodic operating conditions can exist in the system in dependence on the parameters of the system's elements [2]. Simple periodic operating conditions are characterized by the fact that the output signal $\tilde{z}(t)$ of the system to be controlled crosses only one quantization level, as a result of which only a single switching of the scanning direction occurs during an oscillation period. Such operating conditions have been studied in sufficiently great detail by using exact [3, 4] as well as approximate methods [5].

Complex periodic operating conditions occur in the case where the output signal of the system to be controlled crosses more than one quantization level. This results in the fact that additional switchings of the scanning direction occur within a single oscillation period. We shall call such a periodic regime, where the output signal of the system to be controlled crosses N quantization levels, by N-uple complex oscillations. It is readily seen that N is an odd number.

Figure 2 shows the approximate shape of $\tilde{z}(t)$ and $\tilde{x}(t)$ signals when triple oscillations occur in the system. It is obvious from these graphs that the direction of scanning changes three times during the oscillation period T.

In contrast to simple periodic operating conditions, complex operating conditions have not been investigated at all. This obviously explains the fact that they were considered as inoperative regimes, since their amplitude, which determines the scanning loss, has a much higher value than the amplitude of simple oscillations. It will be shown below that such a categorical statement for an entire class of controlled systems is not always valid and that such regimes can be entirely operative in relay extremum control systems (RECS).

The present article is devoted to a study of complex periodic operating conditions, i.e., to the determination of the conditions and the regions of their existence and stability for two cases of approximation of the nonlinear unit's characteristic: a) the function $z = -k_n|y|$ and b) the function $z = -k_n y^2$. The analysis methods used in these two cases differ from each other, since, in approximating the nonlinear units's characteristic by the function $z = -k_n|y|$, the problem is greatly simplified, and the application of the second, more complex, method for obtaining the solution is not advisable.

2. Complex Periodic Operating Conditions in RECS where the Extremum Characteristic is Given by $z = -k_n|y|$

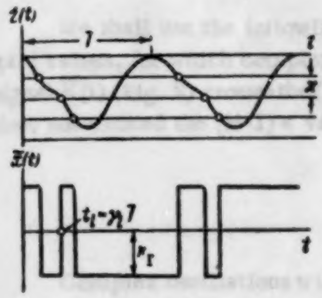

Fig. 2.

From the definition of complex oscillations, it follows that they will take place if the following conditions for the required moments and directions of switchings are satisfied:

$$\tilde{z}(t_i) = -i\varkappa, \quad \dot{\tilde{z}}(t_i) < 0 \qquad (i = 1, 2, \ldots, N), \tag{1}$$

where κ is the magnitude of the quantization level.

Considering that

$$\tilde{z}(t_i) = -k_n|\tilde{y}(t_i)|, \qquad \dot{\tilde{z}}(t_i) = -k_n\dot{\tilde{y}}(t_i)\operatorname{sign}\tilde{y}(t_i),$$

equation (1) can be rewritten in the following form:

$$\tilde{y}(t_i) = i\frac{\varkappa}{k_n}, \qquad \dot{\tilde{y}}(t_i) > 0, \qquad \tilde{y}(t_i) = -i\frac{\varkappa}{k_n}, \qquad \dot{\tilde{y}}(t_i) < 0. \tag{2}$$

Conditions (1) and (2) are equivalent, and the unknown moments of switching t_i that these conditions determine are the same. The difference between (1) and (2) consists in the fact that, since $z(y)$ is an even function, $\tilde{z}(t)$ has a frequency which is twice as large as that of $\tilde{y}(t)$.

Let us introduce the new variable z^0:

$$z^0 = -k_n y.$$

Then, (2) will assume the following form:

$$\tilde{z}^0(t_i) = -i\varkappa, \qquad \dot{\tilde{z}}^0(t_i) < 0, \qquad \tilde{z}^0(t_i) = i\varkappa, \qquad \dot{\tilde{z}}^0(t_i) > 0. \tag{3}$$

Hence, it is obvious that, for determining the frequency and the additional moments of switching, we can consider a relay system which is equivalent to the actual extremum system with respect to the shape, amplitude, and frequency of the signal $\tilde{y}(t)$.

In this system, the nonlinear unit of the system to be controlled is replaced by an amplifier with the gain k_n, the output of which is acted upon by the signal $\tilde{z}^0(t)$, while the extremum regulator is replaced by a relay element with many actuation levels.

It remains to express $\tilde{z}^0(t)$ in explicit form in terms of the system's parameters. It is obvious from Fig. 2 that the signal $\tilde{x}(t)$ can be represented in the form of a sum of elementary components $\tilde{x}_i(t)$, which have the following properties: a) The pulse height and the period of these components are equal to the pulse height and the period of the $\tilde{x}(t)$ signal; b) the $\tilde{x}_i(t)$ components are shifted with respect to the initial point of measurements by the time $t_i = (-1)^i\gamma_i T$, which determines the corresponding switching moment; thus, in changing over from time functions to Laplace image functions, we obtain

$$X(p) = X_1(p)\sum_{i=1}^{N}(-1)^i e^{-\gamma_i Tp} + \sum_{i=1}^{N}(-1)^{i+1}\frac{1 - e^{-\gamma_i Tp}}{p},$$

where $X_1(p)$ is the image of the first elementary component $\tilde{x}_1(t)$ of the $\tilde{x}(t)$ signal, for which $t_1 = \gamma_1 = 0$.

This relationship can be written in a different, more convenient, form:

$$X(p) = X_N(p) \sum_{i=1}^{N} (-1)^i e^{-(\gamma_N - \gamma_i)Tp} + \sum_{i=1}^{N} (-1)^{i+1} \frac{1 - e^{-(\gamma_N - \gamma_i)Tp}}{p},$$

where $X_N(p)$ is the image of the last component of the $\widetilde{x}(t)$ signal.

Hence,

$$Z^0(p) = W(p)\, k_n X_N(p) \sum_{i=1}^{N} (-1)^{i+1} e^{-(\gamma_N - \gamma_i)Tp}$$

$$+ W(p)\, k_n \sum_{i=1}^{N} (-1)^{i+1} \frac{1 - e^{-(\gamma_N - \gamma_i)Tp}}{p},$$

where $W(p) = W_e(p)\, W_{fe}(p)$.

This equation shows that $Z^0(p)$ consists of two components:

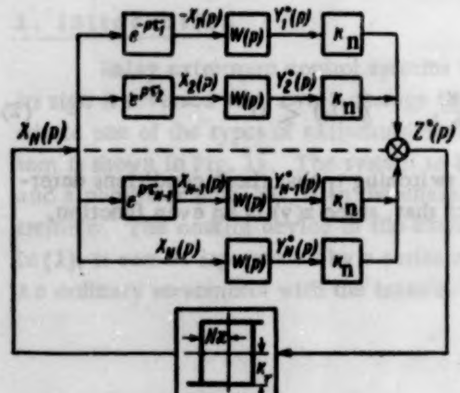

Fig. 3.

a) the image of the periodic component $\widetilde{z}^0(t)$,

$$\widetilde{Z}^0(p) = W(p)\, k_n X_N(p) \sum_{i=1}^{N} (-1)^i e^{-(\gamma_N - \gamma_i)Tp}; \tag{3a}$$

b) the image of the nonperiodic component,

$$Z_a^0(p) = W(p)\, k_n \sum_{i=1}^{N} (-1)^{i+1} \frac{1 - e^{-(\gamma_N - \gamma_i)Tp}}{p}.$$

It follows from Eq. (3a) that, with respect to the periodic component, the relay in the equivalent relay system can have only a single actuation level, which is equal to $N\varkappa$.

Then,

$$X_N(p) = L\{k_r \operatorname{sign}[\widetilde{z}^0(t) \pm N\varkappa]\}.$$

The theoretical block diagram of the equivalent relay system is shown in Fig. 3. In this system, the extremum regulator with many quantization levels is replaced by a relay with hysteresis. Such a relay makes it possible to produce the N-th elementary component of the input signal $\widetilde{x}(t)$. $\widetilde{X}_N(t)$ is supplied to the inputs of the lag elements, the lag time τ_i of which is related to the moments at which the scanning direction is switched, i.e., all the elementary components of the $\widetilde{x}(t)$ signal which act on identical linear elements are obtained at the outputs of the lag elements, after which the components are summed. As a result of this, the signal $\widetilde{y}^0(t)$ is equivalent with respect to frequency, amplitude, and shape to the signal $\widetilde{y}(t)$ in the extremum system.

The unknown switching moments can be determined from the following equations:

$$\widetilde{z}_i^0(t_{N-i}) = N\varkappa, \quad \dot{\widetilde{z}}_i^0(t_{N-i}) > 0, \quad \widetilde{z}_i^0(t_{N-i}) = -N\varkappa, \quad \dot{\widetilde{z}}_i^0(t_{N-i}) < 0,$$

or, by using the relay system's characteristics [6], we can write

$$\operatorname{Im} I_i(\omega) = -N\varkappa, \qquad \operatorname{Re} I_i(\omega) < 0 \qquad (i = 1, 2, \ldots, N). \tag{4}$$

These equations can also be used for determining the frequency of the complex operating conditions and all the switching moments within a single oscillation half-period if the system's parameters are known.

If it is necessary to determine only the oscillation frequency, it is sufficient to consider the last of the conditions (4), i.e.,

$$\operatorname{Im} I_N(\omega) = -N\varkappa, \qquad \operatorname{Re} I_N(\omega) < 0.$$

This equation makes it possible approximately to determine the frequency of self-oscillations, since it does not take into account the additional switchings.

We shall use the following principles for determining the region of the parameters, in this case, the system's gain values, for which complex N-uple oscillations occur. Complex N-uple oscillations will arise if the output signal $\tilde{z}(t)$ (Fig. 2) crosses the (N-2)-th quantization level and if, in this, its maximum deviation from the zero level does not exceed the $(N-1)\kappa$ value, i.e.,

$$\begin{aligned} &\operatorname{Im} I_N(\omega) = -(N-2)\varkappa, \qquad \operatorname{Re} I_N(\omega) < 0, \\ &\max \tilde{z}^0(t) = (N-1)\varkappa. \end{aligned} \tag{5a}$$

Complex oscillations will cease if the output signal $\tilde{z}(t)$ crosses the N-th quantization level and if its maximum deviation from the zero level does not exceed the $(N+1)\kappa$ value, i.e.,

$$\begin{aligned} &\operatorname{Im} I_N(\omega) = -N\varkappa, \qquad \operatorname{Re} I_N(\omega) < 0, \\ &\max \tilde{z}^0(t) = (N+1)\varkappa. \end{aligned} \tag{5b}$$

These equations determine the boundaries of the region of possible gain values for which N-uple complex oscillations exist.

Since it is difficult to determine the maximum deviation of the output quantity of the system to be controlled from the zero level, we shall express it approximately in terms of the first harmonic's amplitude. Then, with an accuracy of the value of the difference $\gamma_{i+1} - \gamma_i \approx 0$, we can write Eqs. (5a) and (5b) in the following form:

$$\begin{aligned} &\operatorname{Im} I_N(\omega) = -(N-2)\varkappa, \qquad \operatorname{Re} I_N(\omega) < 0, \\ &\frac{4}{\pi} k_r k_n |W(j\omega)| = (N-1)\varkappa, \\ &\operatorname{Im} I_N(\omega) = -N\varkappa, \qquad \operatorname{Re} I_N(\omega) < 0, \\ &\frac{4}{\pi} k_r k_n |W(j\omega)| = (N+1)\varkappa. \end{aligned}$$

Each pair of these equations has two unknowns: k_r and ω. The limiting values of the gains k_r and the frequency ω are determined from these equations.

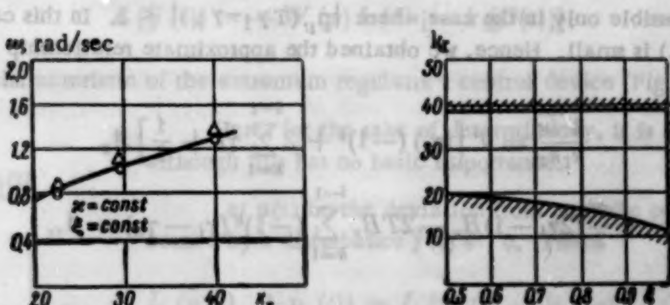

Fig. 4.

The described method was used for investigating triple oscillations in a relay extremum system with a controlled system whose linear unit was represented by an oscillatory element with the transfer function

$$W_0(p) = \frac{k}{T^2p^2 + 2\xi Tp + 1}.$$

For κ = const, we determined the dependence of the frequency of these oscillations on the gain k_r and the region of allowable k_r values for different damping factors ξ for which complex oscillations with the multiplicity N = 3 exist. The theoretical results were confirmed experimentally. Figure 4 shows the corresponding curves (the experimental curve is marked by triangles, and the theoretical curve is marked by points).

3. Complex Periodic Operating Conditions in RECS where the Extremum Characteristic is Given by $z = -k_n y^2$

In the case under consideration, the branches of the extremum characteristic are essentially nonlinear, and, therefore, the method used above is not applicable here. We shall consider another procedure. The equations of the elements entering the system under consideration are given by

$$y = W(p)\,x, \quad W(p) = W_0(p)\,W_{fe}(p) = \frac{P_1(p)}{D_1(p)}, \qquad (6)$$
$$z = -k_n y^2, \quad x = k_r \operatorname{sign}(i\varkappa + z) \quad (i = 1, 2, \ldots, N).$$

By reducing system (6) to its canonic form and by solving it with respect to the intervals $[t_{i-1}, t_i]$ $(i = 1,\ldots,N)$, we obtain the following relationship:

$$\frac{\sqrt{i\varkappa}}{k_r k_n} = T\left[\gamma_i(-1)^i + 2\sum_{k=1}^{i-1}\gamma_k + \frac{1}{2}\right]\frac{P_1(0)}{D_1(0)} + \sum_{\nu=1}^{n}\frac{P_1(p_\nu)}{p_\nu^2 \dot{D}_1(p_\nu)}\left(-\frac{2e^{p_\nu\gamma_i T}}{1+e^{p_\nu T}} - 2\sum_{k=1}^{i-1}(-1)^k e^{p_\nu T(\gamma_i-\gamma_k)} - (-1)^i\right),$$

where p_ν are the bands of the transfer function W(p).

This equation makes it possible to write a system of equations of periods with respect to the unknowns $T\gamma_1$, $T\gamma_1,\ldots,T\gamma_{N-1}$, T. A direct solution of this system of equations is connected with great calculation difficulties, which make this system unsuitable for practical purposes. However, this system can be simplified by means of linear approximations of the $\dfrac{e^{p_\nu\gamma_i T}}{1+e^{p_\nu T}}$ and $e^{p_\nu T(\gamma_i-\gamma_k)}$ functions. By expanding them in a Maclaurin series and by retaining the linear terms, we obtain

$$\frac{e^{p_\nu\gamma_i T}}{1+e^{p_\nu T}} \approx 1 + \frac{2\gamma_i - 1}{4}p_\nu T,$$
$$e^{p_\nu T(\gamma_i-\gamma_k)} \approx 1 + p_\nu T(\gamma_i - \gamma_k).$$

Such an operation is possible only in the case where $|p_\nu(T\gamma_i - \gamma_k)| < 1$. In this case, it is entirely justified, since the difference $(\gamma_i - \gamma_k)$ is small. Hence, we obtained the approximate relationship

$$\frac{\sqrt{i\varkappa}}{k_r k_n} = T\left[(\gamma_i)(-1)^i + 2\sum_{k=1}^{i-1}\gamma_k + \frac{1}{2}\right]A_0 + \frac{T}{2}(2\gamma_i - 1)B_\nu - 2TB_\nu\sum_{k=1}^{i-1}(-1)^k(\gamma_i - \gamma_k) - A_\nu, \qquad (7)$$

where

$$\frac{P_1(0)}{D_1(0)}, \quad B_\nu = \sum_{\nu=1}^{n}\frac{P_1(p_\nu)}{p_\nu \dot{D}_1(p_\nu)}, \quad A_\nu = \sum_{\nu=1}^{n}\frac{P_1(p_\nu)}{p_\nu^2 \dot{D}_1(p_\nu)}.$$

This equation makes it possible to write a system of linear equations for determining $T\gamma_1$, $T\gamma_2,\ldots,T\gamma_{N-1}$, T:

$$
\begin{aligned}
b_{11} &= a_{11}x_1 + \ldots + a_{1N}x_N,\\
b_{21} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N,\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
b_{N1} &= a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{N,N-1}x_{N-1} + a_{NN}x_N,
\end{aligned}
\tag{8}
$$

where $x_k = T\gamma_k (k = 1,\ldots,N-1)$, $x_N = T/2$, b_{i1}, a_{ij} are the coefficients which are obtained from (7) after transformations; $i, j = 1,\ldots, N$.

The system will have a unique solution if its determinant $D \neq 0$.

However, this condition is necessary, but not sufficient for the existence of N-uple oscillations. It may happen that any of the unknowns is negative or equal to zero.

Since

$$x_i = \frac{D_i}{D},$$

for the existence of N-uple oscillations, it is necessary and sufficient that

$$
\begin{aligned}
&1)\ D > 0, \quad D_i > 0,\\
&2)\ D < 0, \quad D_i < 0.
\end{aligned}
$$

By using these conditions, we can determine the parameter regions in which oscillations of a certain multiplicity exist. In particular, it was shown that triple oscillations do not exist in RECS where the transfer function of the linear unit is given by $W(p) = \dfrac{k}{p\,(T^2p + 2\xi Tp + 1)}$, where $\xi = 0.5$, and that simple oscillations are immediately transformed into quintuple oscillations. This was confirmed by experiments.

4. Stability of Complex Periodic Operating Conditions

As was done in [6], we shall determine the stability of complex operating conditions. The operating conditions under consideration are asymptotically stable if the deviation (or variation) $\mu(t)$ from these operating conditions tends to zero with an increase in time for small disturbances. In other words, for analyzing the stability of the periodic conditions, it is necessary to write an equation in terms of variations and to investigate the equilibrium position of the system that is described by this equation.

According to Fig. 5, the value of $\tilde{y}(t)$, which characterizes the periodic operating conditions, is determined by the following equation [if $f(t) = 0$]:

$$L\{\tilde{y}(t)\} = -W(p)\,L\{\Phi\,[-k_n\tilde{y}^2(t)]\},$$

where $\Phi[-k_n\tilde{y}^2(t)]$ is the characteristic of the extremum regulator's control device (Fig. 1b).

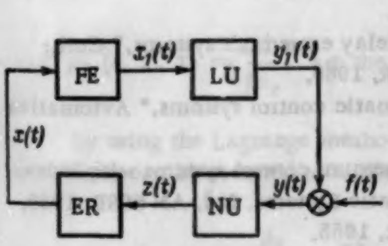

Fig. 5.

Here, for the sake of determinancy, it is assumed that $z = -k_n y^2$, although this has no basic importance.

Let $\mu(t)$ be the deviation from periodic operating conditions which is caused by a disturbance $f(t) \neq 0$. Then,

$$L\{\tilde{y}(t) + \mu(t)\} = L\{f(t)\} - W(p)\,L\{\Phi\,[-k_n(\tilde{y}(t) + \mu(t))^2]\}.$$

Hence, by neglecting small quantities of the second order, we obtain an equation for the deviation $\mu(t)$:

$$L\{\mu(t)\} = L\{f(t)\} - W(p)\,L\{\Phi'\,[-k_n\tilde{y}^2(t)]\,\mu(t)\}. \tag{9}$$

According to [6], we shall have

$$L\{\mu(t)\} = L\{f(t)\} - W(p)L\left\{\sum_{i=1}^{N} k_i \sum_{k=0}^{\infty} \delta(t - kt_i)\,\mu(t)\right\}, \tag{10}$$

where

$$k_i = \frac{2k_p}{|-k_n \tilde{y}^2(t_i)|}.$$

It is obvious from Eq. (10) that a pulse system containing N pulse elements with identical repetition periods, which do not however operate in step, and a continuous element with the transfer function W(p) constitutes the first approximation of a linear extremum control system that operates under conditions of complex oscillations.

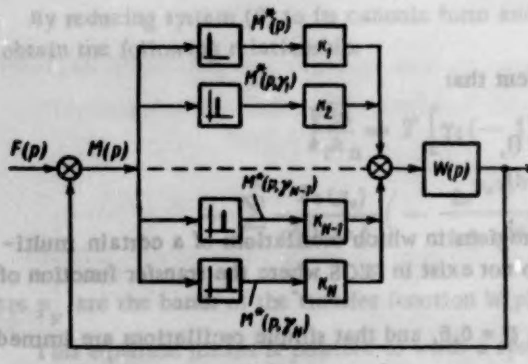

Fig. 6.

The block diagram of the first-approximation system is given in Fig. 6. The stability of the equilibrium position of such a system can be investigated by means of the existing methods.

CONCLUSION

On the basis of our analysis of complex periodic operating conditions, one can provide certain recommendations concerning their application. A completely defined scanning rate, which is characterized by the value k_r, is necessary for an entire class of controlled systems. In this, it may happen that k_r has a value which gives rise to complex oscillatory operating conditions.

Until now, such conditions were, without reservation, defined as inoperative. In order to prevent the occurrence of such operating conditions, the systems were provided with additional logical devices, which made it possible to realize operating switchings through one, two, etc., quantization levels. This led to insignificant increases in frequency, but also to very large increases in amplitude. If the existence of a complex oscillation regime is accepted, one can secure a frequency which is very close to the frequency of the critical operating conditions with a somewhat larger amplitude. The question now arises: Is it better to make the system more complicated by reducing its reliability or to accept somewhat inferior operating conditions, which can, however, be readily realized? It is obvious that, with the exception of certain specific cases, the second method may be chosen, i.e., the complex periodic regime can be used as the operating regime.

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INVESTIGATION OF NONLINEAR UNSTEADY-STATE SYSTEMS WHICH ARE ACTED UPON BY DISCONTINUOUS RANDOM DISTURBANCES

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The present article is concerned with the investigation of nonlinear unsteady-state automatic control systems which are acted upon by discontinuous constraining random and determinate forces. The application of digital, and combined digital and analog computers for determining the distributions of generalized coordinates is considered for the class of automatic control systems under investigation.

1. Statement of the Problem

An arbitrary dynamic system with a finite number n of degrees of freedom can be described by the following Lagrange equations of the second kind under the assumption that all the couplings of this system are independent of time and that they do not contain nonintegrable differential couplings:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k} = Q_k \qquad (k = 1, 2, \ldots, n), \tag{1.1}$$

where $Q_1, \ldots, Q_n$ are the generalized forces which are applied to the system, and $\dot{x}_k = dx_k/dt$ is the generalized velocity of the k-th coordinate.

Assuming that the solutions of system (1.1) without the right-hand sides are known,

$$\begin{aligned} x_k &= x_k(t, c_1, \ldots, c_{2n}) = x_k(t, \bar{c}), \\ \dot{x}_k &= \dot{x}_k(t, c_1, \ldots, c_{2n}) = \dot{x}_k(t, \bar{c}), \end{aligned} \tag{1.2}$$

we obtain, after substituting these solutions in the initial system (1.1)

$$\sum_{i=1}^{2n} \frac{\partial x_k}{\partial c_i}\frac{dc_i}{dt} = 0, \qquad \sum_{i=1}^{2n} \frac{\partial p_k}{\partial c_i}\frac{dc_i}{dt} = Q_k \qquad (k = 1, 2, \ldots, n), \tag{1.3}$$

where $p_k(t, x, \dot{x}) = \dfrac{\partial L}{\partial \dot{x}_k}$ are the generalized pulses.

By using the Lagrange method and by introducing the canonic constants $\alpha_k = c_k$, $\beta_k = c_{n+k}$, the initial system (1.1) can be reduced to a system of 2n first-order differential equations:

$$\frac{d\alpha_s}{dt} = \sum_{k=1}^{n} Q_k \frac{\partial x_k}{\partial \beta_s}, \qquad \frac{d\beta_s}{dt} = -\sum_{k=1}^{n} Q_k \frac{\partial x_k}{\partial \alpha_s} \qquad (s = 1, 2, \ldots, n). \tag{1.4}$$

By performing the above formal transformations, the consideration of the behavior of an actual dynamic system can be reduced to the consideration of the behavior of a material point with finite inertia in 2n-dimensional space. We shall consider below an apparatus which makes it possible to find the distribution function of the

generalized coordinates of a material point in 2n-dimentional space in the case where the point is acted upon by discontinuous random and determinate constraining forces.

2. Derivation of the Distribution Function for the Generalized Coordinates of a Material Inertial Stray Point

It is advisable to start the development of a mathematical apparatus by considering the behavior of an inertialess material point in n-dimensional space for the case where the point is acted upon by discontinuous random constraining forces. The behavior of such a particle is described by Langevin's equation

$$\frac{dx_i}{dt} = \varphi_i \qquad (i = 1, 2, \ldots, n). \tag{2.1}$$

In the case where the input disturbance has the character of white noise, we can write the following expressions for an arbitrary distribution by using the Markov method [1, 2]:

$$W_N(\bar{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\bar{p}\bar{x}) A_N(\bar{p}) d\bar{p},$$
$$A_N(\bar{p}) = \prod_{j=1}^{n} \int_{-\infty}^{\infty} w(x_j) \exp(i\bar{p}x_j) dx_j, \tag{2.2}$$

where p is an arbitrary polar vector.

After transformations, we shall obtain the final expression for the distribution function of the generalized coordinates of a stray Brownian particle under the assumption that the arbitrary initial conditions are given by

$$W(\bar{x}, t) = \prod_{i=1}^{n} \left(\pi \int_0^t D_i(\tau) d\tau\right)^{-1/2} \exp\left\{-\sum \frac{(x_i - x_0)^2}{\int_0^t D_i(\tau) d\tau}\right\}, \tag{2.3}$$

where $D_i(\tau)$ is the power density of white noise in dependence on time.

As an example of finding the coordinate distribution of a stray Brownian particle, we shall consider the one-dimensional case with steady-state noise action under zero initial conditions. In this case, the coordinate distribution function for the Brownian particle will be given by

$$W(x) = \frac{1}{\sqrt{\pi t D}} \exp\left(-\frac{x^2}{Dt}\right). \tag{2.4}$$

We shall assume that the time t varies from 0 to 1. We shall first determine the density of the probability that the particle will be in the interval from x_1 to $x_1 + dx_1$ at the instant of time $t = 1/2$ and that it will be in the interval from x_2 to $x_2 + dx_2$ at the instant of time $t = 1$:

$$P_1 = \frac{1}{\sqrt{\pi D \cdot 1/2}} \exp\left(-\frac{x_1^2}{D \cdot 1/2}\right) \frac{1}{\sqrt{\pi D \cdot 1/2}} \left[-\frac{(x_2 - x_1)^2}{D \cdot 1/2}\right]. \tag{2.5}$$

Then, the probability that the stray particle will be located within certain intervals (a_1, b_1) and (a_2, b_2) at the instants of time $t = 1/2$ and $t = 1$ will be given by the expression

$$P_2 = \int_{a_1}^{b_1} \frac{\exp\left(-\frac{x_1^2}{D \cdot 1/2}\right)}{\sqrt{\pi D \cdot 1/2}} dx_1 \int_{a_2}^{b_2} \frac{\exp\left[-\frac{(x_2 - x_1)^2}{D \cdot 1/2}\right]}{\sqrt{\pi D \cdot 1/2}} dx_2. \tag{2.6}$$

By subdividing the corresponding time interval into ever smaller sections and considering, for each instant of time, the probability that the stray particle will be found in an arbitrary interval (a_n, b_n), we can write

$$P_n = \frac{1}{(D\pi\Delta t)^{n/2}} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \exp\left[-\frac{x_1^2}{Dt_1} - \sum_{j=1}^{n-1} \frac{(x_{j+1}-x_j)^2}{D\Delta t}\right] dx_1 \ldots dx_n. \tag{2.7}$$

Expression (2.7) is the measure of cylindrical multiplicity sets in the space of all functions which are determined by the conditions

$$a_1 < x(t_1) < b_1, \ldots, a_n < x(t_n) < b_n. \tag{2.8}$$

This measure is designated as the Wiener measure. Expression (2.7) makes it possible to describe at any instant of time the behavior of an arbitrary inertialess particle which experiences the action of white noise.

If we let n tend to infinity, we obtain in the limit

$$W(x, t) = N \int_{c_{x_1}} \cdots \int_{c_{x_n}} \exp\left\{-\sum_{k=1}^{n} \frac{[\dot{x}_k(\tau)]^2}{D_k(\tau)} d\tau\right\} dx_1 dx_2 \ldots dx_n. \tag{2.9}$$

We shall now assume that a material particle with finite mass experiences the same constraining force as in Eq. (2.1). Then,

$$\dot{y}_i + B_i(\bar{y}, t) = \varphi_i(t) \qquad (i = 1, 2, \ldots, n). \tag{2.10}$$

On the basis of the equation

$$\dot{x}_i = \dot{y}_i + B_i(\bar{y}, t) \tag{2.11}$$

we can immediately write

$$W(\bar{y}, t) = \int_{c_{y_1}} \cdots \int_{c_{y_n}} \exp\left\{-\int_0^t \sum_{i=1}^{n} \frac{[\dot{y}_i + B_i(\bar{y}, \tau)]^2}{D_i(\tau)} d\tau\right\} F(\bar{y})\, dy_1 dy_2 \ldots dy_n. \tag{2.12}$$

The Jacobian $F(\bar{y})$ of coordinate transformation can be obtained [3] by finding the Fredholm determinant of the linearized Volterra transformation:

$$x_i^\alpha \approx y_i^\alpha + \sum_{\beta=1}^{\alpha} \sum_{k=1}^{n} \frac{\partial B_i(\bar{y}, \Delta\tau\beta)}{\partial y_k} \bar{y}_k^\beta \Delta\tau, \tag{2.13}$$

where $\bar{y}^\beta = \bar{y}(\Delta\tau, \beta)$, and $1 \le \alpha \le N-1$.

Here, N is the number of sections into which the (0, t) interval was divided.

Finally, the distribution function of the generalized coordinates of a stray point with finite mass and n degrees of freedom can be written in the following form for the case of the action of discontinuous constraining forces:

$$\dot{y}_l + B_l(\bar{y}, t) = 0 \quad (1 < l < k). \tag{2.14}$$

$$W(\bar{y}, t) = \int_{c_{y_{k+1}}} \cdots \int_{c_{y_n}} \exp\left\{-\int_0^t \sum_{i=k+1}^{n} \left[\frac{(\dot{y}_i + B_i(\bar{y}, \tau))^2}{D_i(\tau)} - \frac{1}{2}\frac{\partial B_i(\bar{y}, \tau)}{\partial y_i}\right] d\tau\right\} \times dy_{k+1} dy_{k+2} \cdots dy_n. \tag{2.15}$$

Thus, returning to our initial assumptions, we can say that the distribution function of generalized coordinates can be found for an arbitrary dynamic system which experiences the action of discontinuous constraining forces. The development of an engineering method for determining the necessary properties of the system can be reduced to the finding of rational calculation methods that are based on expressions (2.14) and (2.15).

3. Development of the Engineering Method and Aspects of Utilizing Computers

Before considering the method for investigating nonlinear systems, we shall first consider a method for investigating very simple linear closed-loop control systems which contain inertial first-order elements in the forward and the feedback circuits. Such a system can be described by means of two ordinary differential equations:

$$T_1\dot{x} + x = \varphi_n - y, \quad T_2\dot{y} + y = x, \tag{3.1}$$

where φ_n is a random discontinuous signal with the autocorrelation function

$$k_{\varphi_n} = D\delta(t - \tau). \tag{3.2}$$

On the basis of the expressions (2.14) and (2.15), the distribution function of the output coordinate can be represented by the following system:

$$W(x) = N\int_{c_x} \exp\left\{-\int_0^t\left[\frac{1}{D}(T_1\dot{x} + x + y)^2 - \frac{1}{2}\frac{\partial}{\partial x}(x + y)\right]dt\right\}dx \tag{3.3}$$

$$T_2\dot{y} + y = x \quad \text{for} \quad x|_{t=0} = x_0, \quad y|_{t=0} = y_0.$$

The function y is related to x by means of the convolution integral

$$y(t) = \frac{1}{T_2}\int_0^t x(\xi)\exp\left(-\frac{t-\xi}{T_2}\right)d\xi. \tag{3.4}$$

If we take into account the initial conditions used in solving the differential equation in expression (3.3), the distribution function can be written in the form of the following functional:

$$W(x) = N_1\int_{c_x}\exp\left\{-\int_0^t\frac{1}{D}\left[T_1\dot{x} + x + \frac{1}{T_2}\int_0^t x(\xi)\exp\left(-\frac{t-\xi}{T_2}\right)d\xi\right]^2 dt\right\}dx. \tag{3.5}$$

In order to interpret the integral of functional (3.5), we shall write the x function in the following form by using Kotel'nikov's theorem [4]:

$$x(t) = \sum_{-\infty}^{\infty} x\left(\frac{n}{2w}\right)\frac{\sin\pi(2wt - n)}{\pi(2wt - n)}, \tag{3.6}$$

where $w = 1/T$ characterizes the limitation of $x(t)$ with respect to frequency. The important property of the function $(\sin\pi x/\pi x)$ is that it is equal to zero if x is an integer, and that it is equal to unity if $x = 0$. Moreover,

$$\int_{-\infty}^{\infty}\frac{\sin\pi x}{\pi x}dx = 1,$$

$$\int_{-\infty}^{\infty}\frac{\sin\pi(x-r)}{\pi(x-r)}\,\frac{\sin\pi(x-s)}{\pi(x-s)}dx = \begin{cases}1 & \text{for} \quad r = s,\\ 0 & \text{for} \quad r \neq s.\end{cases} \tag{3.7}$$

By substituting expression (3.6) in Eq. (3.5) and taking into account the properties of (3.7), we obtain

$$W(x) = N_1 \int_{-\infty}^{\infty} \cdots \int \exp\Big\{-\frac{\Delta t}{D} \sum_{i=0}^{n-1} \Big[\frac{T_1 x_{i+1} - T_1 x_i}{\Delta t} + x_i + \frac{1}{T_2} \sum_{i=1}^{i} x_i \int_{(i-1)\Delta t}^{i\Delta t} \exp\Big(-\frac{i\Delta t - \xi}{T_2}\Big) d\xi\Big]^2\Big\} dx_1 dx_2 \ldots dx_{n-1}, \tag{3.8}$$

where $\Delta t = T_1/2$.

We can use the following well-known equation for solving (3.8):

$$\int_{-\infty}^{\infty} \cdots \int \exp\Big\{-\sum_{i,j=1}^{n} a_{ij} x_i x_j\Big\} dx_i dx_j = \pi^{n/2} [\det(a_{ij})]^{-1/2}. \tag{3.9}$$

If a computer is to be used for calculations according to (3.9), it is most convenient to use the following algorithm [7]:

$$\det(a_{ij}) = a_{11}^{(n-1)}, \tag{3.10}$$

$$a_{ij}^{(k+1)} = \frac{1}{a_{i+1,\,j+1}^{(k-1)}} [a_{ij}^{(k)} a_{i+1,\,j+1}^{(k)} - a_{i,\,j+1}^{(k)} a_{i+1,\,j}^{(k)}]. \tag{3.11}$$

The calculations can be performed in a somewhat different manner. Actually, the expression for $y(t)$ can be written in the following form:

$$y(t) = \int_0^{(i-1)\Delta t} \frac{1}{T_2} x(\xi) e^{-\frac{(t-\xi)}{T_2}} d\xi + \int_{(i-1)\Delta t}^{i\Delta t} \frac{1}{T_2} x(\xi) e^{-\frac{(t-\xi)}{T_2}} d\xi = y_{i-1} + k_i x_i. \tag{3.12}$$

In this case, expression (3.8) can be rewritten thus:

$$W(x) = N_1 \int_{-\infty}^{\infty} \cdots \int \exp\Big\{-\frac{\Delta t}{D} \times \sum_{i=0}^{n-1} \Big[\frac{T_1 x_{i+1} - T_1 x_i}{\Delta t} + x_i + \sum_{i=1}^{i} (y_{i-1} + k_i x_i)\Big]^2\Big\} dx_1 dx_2 \ldots dx_{n-1}. \tag{3.13}$$

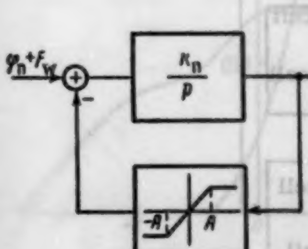

Fig. 1.

The last expression can be written by using simple recurrent formulas:

$$W(x_{i+1}) = N_1 \exp\Big[-\frac{1}{(\Delta t + \sigma_{i-1} D_i} - \frac{(\sigma_{i-1} D_i)^2}{(\sigma_{i-1} D_i + \psi_i)(\Delta t + \sigma_{i-1} D_i)}\Big] x_{i+1}^2,$$

$$\psi_i = \Big[1 - \frac{1}{(\Delta t + \sigma_{i-2} D_{i-1})} - \frac{(\sigma_{i-1}^2 D_{i-1})^2}{(\sigma_{i-2} D_{i-1} + \psi_{i-1})(\Delta t + \sigma_{i-2} D_{i-1})}\Big]. \tag{3.14}$$

We shall consider an example of the exact solution for nonlinear systems. The system (Fig. 1) is acted upon by the sum of a step signal and a white-noise random signal. The block diagram of the program for solving this system by means of a Ural-1 computer is shown in Fig. 2. Figure 3 shows the solution for $\Delta t = 0.25$ (curve 3) and $\Delta t = 0.5$ sec (curve 2) and the histogram (curve 1) which was obtained on the basis of 200 realizations. As an example of how the system's excited state affects the distribution, Fig. 4 shows the distributions of the signal $\underline{x}$ at the

instant of time t = 0.25 (curve 1) and t = 2.25 sec (curve 2). From this figure, it is obvious that the system's excited state considerably affects the distribution. Let us consider another example of a nonlinear closed-loop control system whose forward circuit contains a linear element, while the feedback circuit contains a nonlinear inertial element:

$$T\dot{x} + x = \varphi_n - y, \qquad \dot{y} + y^2 = x. \tag{3.15}$$

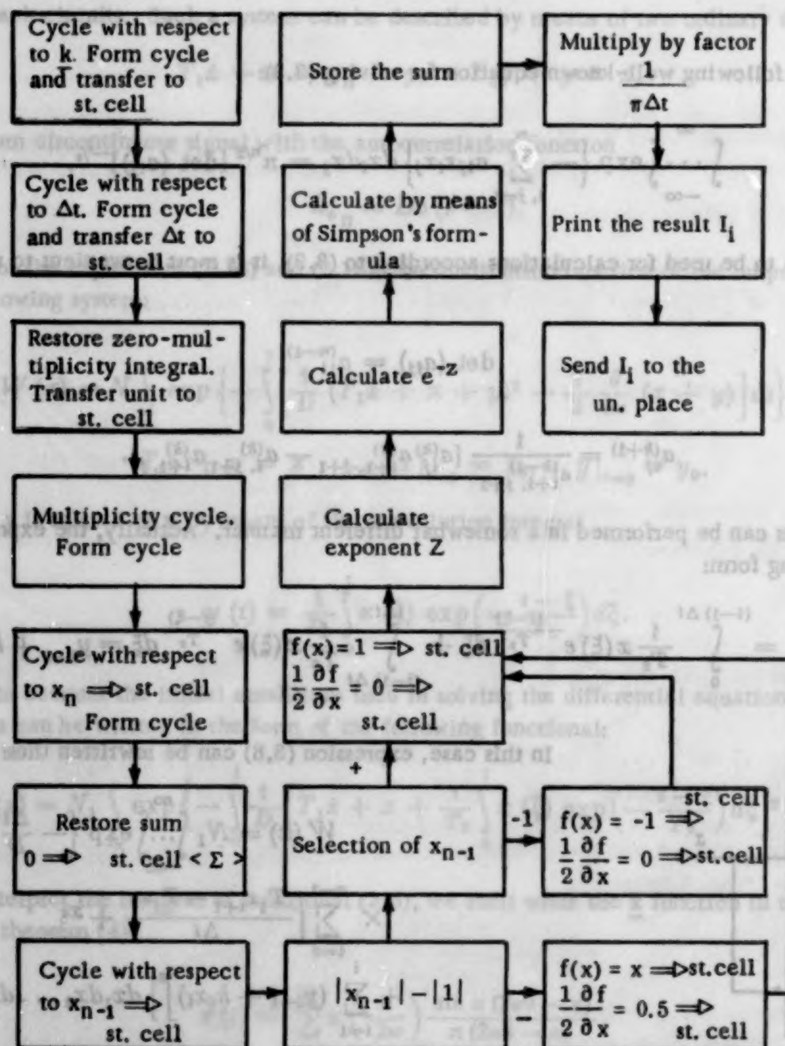

Fig. 2.

In order to find the distribution of the $\underline{x}$ coordinate, it is necessary to know the $y = f(x)$ dependence under the condition that $\underline{x}$ be a constant quantity at the assigned discreteness step. By using the Taylor–Cauchy transform [5], we obtain

$$y_i = y(t_i) \approx (\Delta t - {}^2/_3\, \Delta t^3)\, x_i + {}^2/_{15}\, \Delta t^5 x_i^2 + y_{i-1} = a x_i + b x_i^2 + y_{i-1}. \tag{3.16}$$

The distribution at the system's output will be written thus:

$$W(x) = N_1 \int_{-\infty}^{\infty} \ldots \int \exp\left[-\sum_{i=1}^{n-1} \frac{\Delta t}{D}\left(\frac{T_1 x_{i+1} - T_1 x_i}{\Delta t} + x_i + a x_i + b x_i^2 + y_{i-1}\right)^2\right] dx_i. \tag{3.17}$$

A computer can be used for performing calculations according to expression (3.17). By using the statistical linearization method [6], expression (3.17) can be reduced to quadratic forms. The calculations are performed by using the recurrent formulas (3.14). Actually, expression (3.16) yields a functional inertialess coupling between y_i, y_{i-1}, and x_i, while the linearization is performed by means of the usual method, since the distribution of x_i and y_{i-1} for the inertial systems is sometimes known. Thus, the distribution for $x_1 = x(\Delta t)$ can be written immediately for the initial zero conditions.

$$W(x_1) = N\exp\left(-\frac{T_1^2}{D\Delta t}x_1^2\right). \tag{3.18}$$

By using this expression, we shall find the equivalent transfer constant, which relates x_2 and y_1:

$$k_1 = \frac{1}{2}\left(a + a^2 + {}^3/_2\,\frac{b^2D\Delta t}{T_1^2}\right). \tag{3.19}$$

Knowing k_1, we can find the distribution $W(x_2)$ for $t = 2\Delta t$:

$$W(x_2) =$$

$$N\exp\left\{-\left[\frac{T_1^2}{\Delta t D} - \frac{T_1^2}{D\Delta t}\,\frac{(T_1/\Delta t - 1 - k_1)}{\left[\left(\frac{T_1}{\Delta t} - 1 - k_1\right)^2 + \left(\frac{T_1}{\Delta t}\right)^2\right]}\right\}x_2^2$$

$$= N_1\exp(-\psi_1 x_2^2). \tag{3.20}$$

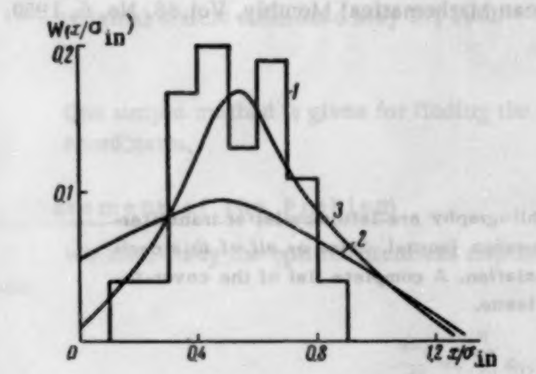

Fig. 3.

The expression for $W(x_3)$ will be written in the following form:

$$W(x_3) = N_1\int\exp - \left\{\frac{\Delta t}{D}\left[\frac{Tx_3}{\Delta t} - \left(\frac{T}{\Delta t} - 1 - k_2\right)x_2 + y_1\right]^2 + \psi_1 x_2^2\right\}dx_2, \tag{3.21}$$

where $k_2 = a + a^2 + {}^3/_2\,\frac{b^2}{\psi_1}$.

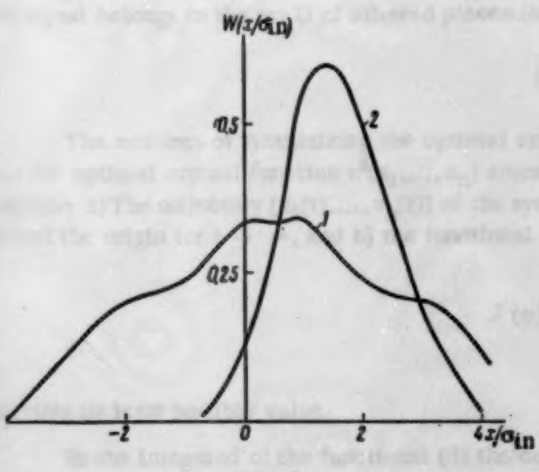

Fig. 4.

All the subsequent steps are realized in a similar manner. It is advisable to investigate complex automatic control systems by using combined circuit diagrams of analog and digital computers. Actually, if the additional conditions (2.15) cannot be solved in the general form, analog or digital computers can be used for solving them. The obtained numerical data are used in calculating the expression (2.14) by means of digital computers by using algorithms which are similar to that given in Fig. 2.

Thus, by using combined computer circuits or by means of digital computers, we can obtain exact or approximate expressions for the coordinate distributions of nonlinear, generally, unsteady-state, mock-ups of complex control systems that operate in the excited state under the action of discontinuous random and determinate input disturbances.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**

SYNTHESIS OF RELAY SYSTEMS FROM THE MINIMUM INTEGRAL QUADRATIC DEVIATION

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One simple method is given for finding the optimal control law in the form of a function of the phase coordinates.

1. Statement of the Problem

We shall study the optimal transient response in a control system that is described by the differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + b_i u \qquad (i = 1, \ldots, n), \tag{1}$$

where x_i are the phase coordinates system, a_{ij}, b_i are constant coefficients. The function which describes the control signal belongs to the set U of allowed piecewise continuous functions that satisfy the condition

$$|u(t)| \leqslant 1. \tag{2}$$

The problem of synthesizing the optimal control law is formulated as follows. For the system (1) we must find the optimal control function $u^0(x_1, \ldots, x_n)$ among all allowed control laws which is such that two conditions are satisfied: a) The trajectory $[x_1(t), \ldots, x_n(t)]$ of the system (1) starts from any initial position $[x_1(0), \ldots, x_n(0)]$ and tends toward the origin for $t \to \infty$, and b) the functional

$$J(u) = \int_0^\infty \Big(\sum_{i=1}^{n} a_i x_i^2\Big) dt \tag{3}$$

acquires its least possible value.

In the integrand of the functional (3) the coefficients a_i are positive and certain of them may be equal to zero. Therefore the quadratic form $V = \sum_{i=1}^{n} a_i x_i^2$ is positive with a constant sign.

The optimal control function $u^0(x_1, \ldots, x_n)$ can be found from the L. S. Pontriagin principle of the maximum [1]. It is precisely the optimal control law u^0 in the form of a time function that is defined as

$$u^0(t) = \operatorname{sign}\Big(\sum_{i=1}^{n} b_i \psi_i(t)\Big).$$

Here $\psi_i(t)$ $(i = 1, \ldots, n)$ are solutions of the conjugate system

$$\frac{d\psi_i}{dt} = -\sum_{j=1}^{n} a_{ji}\psi_j \qquad (i = 1, \ldots, n)$$

for the initial conditions $\psi_i(0)$ which were chosen so that the trajectory of the system (1) tends toward the origin. It is evident that the values $\psi_i(0)$ depend on the initial conditions $x_i(0)$. If we could find the relationship between $\psi_i(0)$ and $x_i(0)$ namely $\psi_i(0) = \varphi_i[x_1(0),\dots,x_n(0)]$ which assures passage of the system trajectory through the origin, then we would obtain the optimal control function

$$u^0(x_1,\dots,x_n) = \operatorname{sign}\left[\sum_{i=1}^{n} b_i\varphi_i(x_1,\dots,x_n)\right].$$

The difficulty of this method resides in particular in finding the analytical relationships between $\psi_i(0)$ and $x_i(0)$.

The problem which we have posed can also be solved by the method of dynamic programming. Using the Bellman method [2], the optimal control law $u^0(x_1,\dots,x_n)$ is determined from the following partial differential equation:

$$\min_{u\in U}\left[\sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}x_j + b_i u\right)\right] = 0. \tag{4}$$

From this we find the optimal control law

$$u^0(x_1,\dots,x_n) = -\operatorname{sign}\left(\sum_{i=1}^{n} b_i\frac{\partial f}{\partial x_i}\right). \tag{4a}$$

Here the difficulty resides in finding the function $f(x_1,\dots,x_n)$ from the partial differential equation (4) in which the form (4a) has been substituted for u.

In this paper we cite a simple method for finding the function $u^0(x_1,\dots,x_n)$ which is based on a certain Lyapunov theorem.

2. The Solution of the Problem

At first we shall make one assumption relative to the system (1). Assume that the system is intrinsically stable (i.e., the roots of the characteristic equation $a_{ij} - \lambda\delta_{ij}| = 0$ have only negative real parts). Then in accordance with the Lyapunov theorem as generalized by Yu. I. Alimov [3] there exists one and only one definitely positive quadratic form $W(x_1,\dots,x_n) = \sum_{i,j=1}^{n} A_{ij}x_i x_j$ for the specified quadratic form $V = \sum_{i=1}^{n} a_i x_i^2$ that has a positive sign; this form satisfies the equation

$$\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}\sum_{j=1}^{n} a_{ij}x_j = -\sum_{i=1}^{n} a_i x_i^2. \tag{5}$$

From this it follows that the coefficients A_{ij} $(i, j = 1,\dots,n)$ are solutions of the following system of algebraic equations:

$$\sum_{j=1}^{n}(A_{ij}a_{jk} + A_{kj}a_{ji}) = -a_i\delta_{ik} \qquad (i, k = 1,\dots,n),$$

where δ_{ik} is the Kronecker symbol.

Now we assume a priori that there exists an optimal control law $u^0(x_1,\dots,x_n)$ that transfers the initial point to the origin. Differentiating the derived function $W = \sum_{i,j=1}^{n} A_{ij}x_i x_j$ along the integral curve for the system (1), we obtain

$$\frac{dW}{dt}=\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}\Big(\sum_{j=1}^{n}a_{ij}x_j+b_iu\Big)=-\sum_{i=1}^{n}a_ix_i^2+u\sum_{i=1}^{n}b_i\frac{\partial W}{\partial x_i} \tag{6}$$

in view of Eq. (5).

Integrating both sides of Eq. (6), we obtain

$$\int_0^\infty\Big(\sum_{i=1}^{n}a_ix_i^2\Big)dt=-\int_0^\infty dW+\int_0^\infty u\Big(\sum_{i=1}^{n}b_i\frac{\partial W}{\partial x_i}\Big)dt=W(0)-W(\infty)+\int_0^\infty u\Big(\sum_{i=1}^{n}b_i\frac{\partial W}{\partial x_i}\Big)dt, \tag{7}$$

where $W(0)=W[x_1(0),\ldots,x_n(0)]$, $W(\infty)=W[x_1(\ \ ),\ldots,x_n(\infty)]$.

Since the function W has a definite sign and an optimal control law exists, the value of $W(\infty)$ goes to zero.

From (7) it follows that in order for the functional (3) to be minimal with respect to u it is necessary and sufficient that the optimal control law be defined in the following manner (cf. Appendix):

$$u^0(x_1,\ldots,x_n)=-\operatorname{sign}\Big(\sum_{i=1}^{n}b_i\frac{\partial W}{\partial x_i}\Big)=-\operatorname{sign}\Big[\sum_{i=1}^{n}b_i\sum_{j=1}^{n}(A_{ij}+A_{ji})x_j\Big]. \tag{8}$$

On the other hand the function $W=\sum_{i,j=1}^{n}A_{ij}x_ix_j$ is a Lyapunov function for the system (1) with the control law (8). Namely,

$$\frac{dW}{dt}=-\sum_{i=1}^{n}a_ix_i^2-\Big|\sum_{i=1}^{n}b_i\frac{\partial W}{\partial x_i}\Big|<0.$$

3. On a Slippage Mode

In this section we shall demonstrate that in the system (1) equipped with the optimal control law (8) there exists a slippage mode. For convenience in our investigation we shall introduce the variable x_{n+1}:

$$x_{n+1}=\sum_{i=1}^{n}b_i\frac{\partial W}{\partial x_i}=\sum_{i=1}^{n}\sum_{j=1}^{n}2b_iA_{ij}x_j.$$

We shall assume that $A_{ij}=A_{ji}$. Then we can write the system (1) in the form

$$\frac{dx_i}{dt}=\sum_{j=1}^{n}a_{ij}x_j-b_i\operatorname{sign}x_{n+1}\qquad(i=1,\ldots,n),$$
$$\frac{dx_{n+1}}{dt}=\sum_{j=1}^{n}a_{n+1,j}x_j-b_{n+1}\operatorname{sign}x_{n+1}, \tag{9}$$

where

$$a_{n+1,j}=\sum_{i=1}^{n}\sum_{k=1}^{n}2b_ia_{nj}A_{ik},\qquad b_{n+1}=\sum_{i=1}^{n}\sum_{j=1}^{n}2A_{ij}b_ib_j. \tag{10}$$

From (9) it is easy to see [4] that if $b_{n+1}>0$ it follows that the system trajectory entering the region

$$\left|\sum_{j=1}^{n} a_{n+1,\,j}x_j\right| < b_{n+1},$$

will definitely land on the hyperplane

$$\sum_{i=1}^{n}\sum_{j=1}^{n} 2b_iA_{ij}x_j = 0$$

in the space $(x_1,\dots,x_n)$. Then, without leaving that hyperplane, the image point will move along a trajectory determined by the equation

$$\frac{dx_i}{dt} = \sum_j a_{ij}x_j - \frac{b_i}{b_{n+1}}\sum_j a_{n+1,\,j}x_j \quad (i = 1,\dots,n)$$

until it lands at the origin.

Therefore in order to assert the existence of a slippage mode it is sufficient to demonstrate that $b_{n+1} > 0$ for any specified coefficients b_i $(i = 1,\dots,n)$.

In fact, b_{n+1} is the quadratic form (10) relative to the parameters b_i with the coefficients A_{ij} $(i, j = 1,\dots,n)$. It is evident that b_{n+1} will always be positive.

4. The Synthesis of Pulse Systems

Assume that in the system (1) the control function $\underline{u}$ is a pulse function with amplitude modulation and equal repetition periods; i.e., the system is a pulse system of the first type [5]. Because of this property of the pulse function $\underline{u}$ it is not difficult to transform the system (1) into a system of difference equations with constant coefficients

$$x_{i,\,m+1} = \sum_{j=1}^{n} p_{ij}x_{jm} + c_iu_m \quad (i = 1,\dots,n) \tag{11}$$

(where p_{ij}, c_i are constant numbers which depend on the repetition periods). Here the symbols x_{im}, u_m are an abridged notation for the variables x_i, $\underline{u}$ treated at the instant $m\tau$; i.e., $x_{im} = x_i(m\tau)$, $u_m = u(m\tau)$, where τ is the repetition period and $\underline{m}$ acquires integer values 0,1,2,... In order to perform the subsequent analysis in more abbreviated form, we shall make use of the matrix form for the system (11); this system can then be written as

$$x_{m+1} = Px_m + Cu_m. \tag{12}$$

The synthesis problem is formulated as follows. For any initial conditions x_{i0} $(i = 1,\dots,n)$ it is required to find the series of controlling quantities $\{u_m\}$ $(m = 0,1,2,\dots)$ with the limitation

$$|u_m| \leqslant 1, \tag{13}$$

which are such that the functional

$$J(u_m) = \sum_{m=0}^{\infty}\left(\sum_{i=1}^{n} a_ix_{im}^2\right), \quad a_i > 0 \tag{14}$$

has its minimum value and $\lim\limits_{m\to\infty} x_{im} = 0$ $(i = 1,\dots,n)$.

The problem can easily be solved by the method that is used for analog control systems.

Assume that the moduli of the eigenvalues λ_i for the matrix P are less than unity. Then the following theorem [6] will be valid (the situation is analogous that corresponding to analog systems). For any specified definitely-positive quadratic form $V_m = x'_mAx_m$ there exists a single definitely-positive quadratic form $W_m = x'_mBx_m$ which on the basis of the system $x_{m+1} = Px_m$ satisfies the equation

$$W_{m+1} - W_m = -V_m.$$

The matrix B is defined by the equation

$$P'BP - B = -A.$$

Assume the matrix $A = \| A_{ij} \|$, where $A_{ij} = a_i \delta_{ij}$. Then the functional (14) can be written as

$$J(u_m) = \sum_{m=0}^{\infty} x'_m A x_m = \sum_{m=0}^{\infty} V_m.$$

Now we shall compute the difference $W_{m+1} - W_m$. In view of the system (12),

$$\begin{aligned} W_{m+1} - W_m &= (x'_m P' + C' u_m)\, B\, (P x_m + C u_m) - x'_m B x_m \\ &= -x'_m A x_m + (C'BP x_m + x'_m P'BC)\, u_m + (C'BC)\, u_m^2. \end{aligned} \tag{15}$$

Summing both parts, we find

$$J(u_m) = \sum_{m=0}^{\infty} V_m = W_0 + \sum_{m=0}^{\infty} [(C'BP x_m + x_m P'BC)\, u_m + (C'BC)\, u_m^2].$$

From this it follows that when the limitation (13) is taken into account the functional $J(u_m)$ (14) will be minimal with respect to u_m only in the case where u_m is defined as follows:

$$\begin{aligned} u_m &= \frac{1}{2(C'BC)} (C'BP x_m + x'_m P'BC) && \text{for } \left| \frac{1}{2(C'BC)} (C'BP x_m + x'_m P'BC) \right| < 1, \\ u_m &= 1 && \text{for } \frac{1}{2(C'BC)} (C'BP x_m + x'_m P'BC) \leqslant -1, \\ u_m &= -1 && \text{for } \frac{1}{2(C'BC)} (C'BP x_m + x'_m P'BC) \geqslant 1. \end{aligned} \tag{16}$$

Substituting (16) into Eq. (11), we shall obtain the difference $W_{m+1} - W_m$, which will always be negative; this will guarantee the asymptotic stability of the system (11).

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Example

As an example we shall study the following second order equation:

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + x = u, \quad 0 < \alpha < 1, \quad |u| \leqslant N. \tag{17}$$

The functional which is minimized is written as

$$J(u) = \int_0^{\infty} x^2 dt.$$

We shall rewrite (17) in the normal form

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1 - 2\alpha x_2 + u.$$

In accordance with the Lyapunov theorem, for a specified $V = x^2$ there exists $W = A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2$ which satisfies the equation

$$\frac{\partial W}{\partial x_1} x_2 + \frac{\partial W}{\partial x_2}(-x_1 - 2\alpha x_2) = -x_1^2.$$

Comparing both parts of this equation, we find

$$A_{11} = \alpha + \frac{1}{4\alpha}, \quad A_{12} = \frac{1}{2}, \quad A_{22} = \frac{1}{4\alpha}.$$

Therefore the optimal control law is of the following form:

$$u^0 = -N \operatorname{sign}\left(\frac{\partial W}{\partial x_2}\right) = -N \operatorname{sign}\left(x_1 + \frac{1}{2\alpha} x_2\right) = -N \operatorname{sign}\left(x + \frac{1}{2\alpha}\frac{dx}{dt}\right).$$

It is easy to show that for such a control law there always exists a slippage mode [7].

Appendix

In general the control law in the form (8) which derives directly from (7) does not yield the minimal value of the integral, since in the first part of Eq. (7) the integrand function depends not only on the control law $\underline{u}$ but also on the functions $x_i(t)$ which are defined by the control law $\underline{u}$ proper. However, it can be demonstrated that in the case under study the optimal control law is indeed determined from formula (8). For this purpose we shall examine the relationship between the Lyapunov function $W(x_1,\ldots,x_n)$ and the function $f(x_1,\ldots,x_n)$ which is defined by the functional Bellman equation:

$$\sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sum_{j=1}^{n} a_{ij} x_j = \left|\sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i}\right| = 0. \tag{18}$$

It is easy to prove that the function $f(x_1,\ldots,x_n)$ which satisfies the condition

$$\frac{\partial f}{\partial x_i} = \lambda(x_1,\ldots,x_n)\frac{\partial W}{\partial x_i} \quad (i = 1,\ldots,n), \tag{19}$$

with

$$\lambda(x_1,\ldots,x_n) = \frac{\sum_{i=1}^{n} a_i x_i^2}{\sum_{i=1}^{n} a_i x_i^2 + \left|\sum_{i=1}^{n} b_i \frac{\partial W}{\partial x_i}\right|} > 0,$$

will be a solution of Eq. (18). In fact, Eq. (18) becomes an identity when Eq. (19) is substituted into it. It remains for us to demonstrate that the function $f(x_1,\ldots,x_n)$ which satisfies condition (19) is definitely-positive. Multiplying both sides of Eq. (19) by x_i and summing the results, we obtain

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = \lambda(x_1,\ldots,x_n) \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} x_i = 2\lambda(x_1,\ldots,x_n) W(x_1,\ldots,x_n).$$

It is evident that the sum

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i \tag{20}$$

is rigorously positive and goes to zero only at the origin. Note that the sum (20) is the scalar product of the vector-radius drawn from the origin to some point $\underline{a}$ on the surface $f(x_1,\ldots,x_n) = c$ ($\underline{c}$ is a constant number) and the vector-normal to that surface which passes through the point $\underline{a}$. Assume the equations $f(x_1,\ldots,x_n) = c_1$, $f(x_1,\ldots,x_n) = c_2$ defined two surfaces F_1 and F_2 in the phase space $(x_1,\ldots,x_n)$. Because the sum (20) is rigorously positive we conclude that the surfaces F_1 and F_2 are convex and that F_1 is located completely within F_2 if $c_1 < c_2$. Since the

function $f(x_1,\dots,x_n)$ is equal to zero at the origin it follows that $f(x_1,\dots,x_n)$ is definitely positive. Since the multiplier $\lambda(x_1,\dots,x_n)$ does not change sign at any point, we can write

$$u^0 = -\operatorname{sign}\left(\sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i}\right) = -\operatorname{sign}\left[\lambda(x_1,\dots,x_n)\sum_{i=1}^{n} b_i \frac{\partial \overline{W}}{\partial x_i}\right] = -\operatorname{sign}\left(\sum_{i=1}^{n} b_i \frac{\partial \overline{W}}{\partial x_i}\right).$$

Our proposition has been proved.

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ON THE EXACT DETERMINATION OF PERIODIC MODES IN A RELAY AUTOMATIC CONTROL SYSTEM WITH SEVERAL RELAY ELEMENTS

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The paper studies an exact method for determining periodic modes in a relay automatic control system that is designed according to an arbitrary block diagram with any number of relay elements. The problem is reduced to formulating the equations for the periods. The results are expressed either in the form of complete Fourier series in terms of the coefficients of the original system of differential equations, or in closed form using the roots of the characteristic equation.

Heretofore numerous and varied papers have made a detailed and all-round study of relay automatic control systems with one relay element. In contrast to this little attention has heretofore been devoted to relay systems with several relay elements. In view of the complexity of the general problems which involve the dynamics of systems with several nonlinear elements (including relay elements) particular problems which are simpler but of practical importance have been isolated from these general problems and analyzed (for example, the determination of periodic movements and the investigation of their stability).

In order to solve these problems approximate methods and techniques were used which were based on the principle of harmonic balance. It should be said the results obtained in this way make it possible in a number of cases to make judgements concerning the general dynamic properties and the pattern of movements of a nonlinear system.

However, the approximations permitted in these methods do not permit the full exposure of certain phenomena in nonlinear systems; in the case where several nonlinearities are present the discrepancies between the approximate results and the actual properties of the system prove to be appreciable. Therefore those methods that make it possible to achieve exact or almost exact results are of great significance.

Such methods were proposed in [1] for a relay system with two relays that was designed according to a single-loop network, and in [2] for an arbitrary system with two symmetrical relays.

In this paper we give the exact solution of the problem involving periodic movements in a relay automatic control system with an arbitrary block diagram containing any (finite) number of relay elements with arbitrary characteristics.

1. The Transformation of the Original Equations of Motion

We shall study the system of original differential equations of motion

$$\dot{x}_k = \sum_{\alpha=1}^{n} a_{k\alpha}x_\alpha + \sum_{\beta=1}^{m} b_{k\beta}f_\beta(\sigma_\beta) \quad (k = 1, 2, \ldots, n), \tag{1a}$$

$$\sigma_\beta = \sum_{\gamma=1}^{n} c_{\beta\gamma}x_\gamma \quad (\beta = 1, 2, \ldots, m), \tag{1b}$$

where $a_{k\alpha}$, $b_{k\beta}$, $c_{\beta\gamma}$ are constants, $f_\beta(\sigma_\beta)$ are as yet arbitrary nonlinear functions of their arguments, $\underline{n}$ is the order of the system, $\underline{m}$ is the number of nonlinearities.

Assume*

$$N(p) = |\delta_{k\alpha}p - a_{k\alpha}| = p^n + \nu_1 p^{n-1} + \ldots + \nu_n \quad \left(p \equiv \frac{d}{dt}\right) \tag{2}$$

is the characteristic polynomial for the system (1).

Replacing the s-th column of the determinant N(p) in turn by the columns $b_{k\beta}$ for all $\beta = 1,2,\ldots,m$ and $s = 1,2,\ldots,n$, we must obtain mn new differentiation operators whose order is no greater than (n − 1):

$$M_s^\beta(p) = \sum_{k=1}^{n} b_{k\beta} N_{ks}(p) = \mu_{s1}^\beta p^{n-1} + \ldots + \mu_{sn}^\beta \tag{3}$$
$$(\beta = 1, 2, \ldots, m; \; s = 1, 2, \ldots, n).$$

Here $N_{ks}(p)$ is the algebraic complement of the element in the k-th row and the s-th column of the determinant N(p).

Using the operators $M_s^\beta(p)$ and the coefficients $c_{\beta\gamma}$, we form m^2 linear forms of the type

$$M_{\beta\delta}(p) = \sum_{\gamma=1}^{n} c_{\beta\gamma} M_\gamma^\delta(p) = \mu_{\beta\delta 1} p^{n-1} + \ldots + \mu_{\beta\delta n} \quad (\beta, \delta = 1, 2, \ldots, m). \tag{4}$$

Making use of the operators (2) and (4), the original equations of motion can be reduced to the form**

$$N(p)\, y_\beta = f_\beta(\sigma_\beta), \tag{5a}$$

$$\sigma_\beta = \sum_{\delta=1}^{m} M_{\beta\delta}(p)\, y_\delta \quad (\beta = 1, 2, \ldots, m). \tag{5b}$$

Here y_β are new variables that are related to the original variable x_k by the differential relationships

$$x_k = \sum_{\delta=1}^{m} M_k^\delta(p)\, y_\delta. \tag{6}$$

In accordance with Eq. (5a) the new variables y_β and their derivatives up to the (n − 1)-th inclusive are continuous time functions. The derivative $y_\beta^{(n)}$ is subject to discontinuities only in the case where the corresponding nonlinear characteristic $f_\beta(\sigma_\beta)$ is discontinuous; the discontinuities of $f_\beta(\sigma_\beta)$ are transferred without variations to $y_\beta^{(n)}$.

As far as the arguments σ_β of the nonlinear characteristics are concerned, it follows that they are continuous time functions since the order of all the operators $M_{\beta\delta}(p)$ in (5b) is no higher than the (n − 1)-th order.

The derivative $\dot{\sigma}_\beta$ can prove to be discontinuous at the points of discontinuity*** $f_\delta(\sigma_\delta)$, as long as the order of operator $M_{\beta\delta}(p)$ is not lower than (n - 1). Therefore, in the case of the discontinuous characteristics $f_\delta(\sigma_\delta)$, (it is this case which is investigated below) their "switching" may prove to be of the "slippage" type for certain initial conditions.

We shall limit ourselves to determining those periodic movements which consist completely of normal (non-slippage) "switchings" of nonlinear characteristics.

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* $\delta_{k\alpha}$ is the Kronecker symbol.

** The equivalence of the differential equations (1) and (5) for (6) is proved in the Appendix, section 1. For a system with one nonlinearity [6] cites a different, more complex, proof of the equivalence of these equations.

*** Here they are in the form of nonlinear characteristics ($\delta = 1,2,\ldots,m$), which can cause discontinuity at the derivative $\dot{\sigma}_\beta$.

2. The Search for Periodic Solutions in the Form of Complete Fourier Series for Eq. (5) When the Characteristics are Piecewise-Linear

We shall seek periodic solutions of Eq. (5) for the case of piecewise-linear characteristics on the assumption that the type of periodic solutions is specified. This means that the sequence of "switching" is known for each nonlinear characteristic separately. The "switching" sequence for different nonlinear characteristics is not stipulated.

Assume that the sequence of "switching" instants $t_{\beta j_\beta}$ is specified for the characteristic $f_\beta(\sigma_\beta)$ over the period of movement T by the inequalities

$$t_{\beta 0} < t_{\beta 1} < \ldots < t_{\beta N_\beta} = t_{\beta 0} + T. \tag{7}$$

Then the nonlinear characteristics are determined by the

$$f_\beta(\sigma_\beta) = \bar{a}_{\beta j_\beta}\sigma_\beta + \bar{b}_{\beta j_\beta} \quad (t_{\beta j_\beta - 1} < t < t_{\beta j_\beta})$$
$$(\beta = 1, 2, \ldots, m; \quad j_\beta = 1, 2, \ldots, N_\beta), \tag{8}$$

where $\bar{a}_{\beta j_\beta}$, $\bar{b}_{\beta j_\beta}$ are unknown numbers among which we may find numbers that are equal for different subscripts j_β; N_β is the over-all number of transitions from one linear segment of the nonlinear characteristic to another during the period of movement for the characteristic $f_\beta(\sigma_\beta)$.

Equations (8) alternate for fixed β at discrete instants $t_{\beta j_\beta}$, when the corresponding argument σ_β first reaches the specified value $\sigma_{\beta_{j_\beta} sp}$, i.e.,

$$\sigma_\beta(t_{\beta j_\beta}) = \sigma_{\beta j_\beta \, sp} \, . \tag{9}$$

If we introduce (8) into (5), then we obtain

$$L_{\beta j_\beta}(p)\, y_\beta(t) = \bar{b}_{\beta j_\beta} + \bar{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' M_{\beta\delta}(p)\, y_\delta(t) \quad (t_{\beta j_{\beta-1}} < t < t_{\beta j_\beta})$$
$$(\beta = 1, 2, \ldots, m; \; j_\beta = 1, 2, \ldots, N_\beta), \tag{10}$$

where

$$L_{\beta j_\beta}(p) = N(p) - \bar{a}_{\beta j_\beta} M_{\beta\beta}(p) = p^n + \lambda_{\beta j_\beta 1} p^{n-1} + \ldots + \lambda_{\beta j_\beta n},$$
$$\lambda_{\beta j_\beta q} = \nu_q - \bar{a}_{\beta j_\beta} \mu_{\beta\beta q} \quad (q = 1, 2, \ldots, n).$$

Here and throughout, the prime associated with the summation sign in (10) denotes that the term $\delta = \beta$ is omitted for summation.

We shall introduce the periodic pulse time functions $y_{\beta j_\beta}(t)$ with the period T, that are determined by the equations

$$y_{\beta j_\beta}(t) = 0 \quad \text{for} \quad \nu T + t_{\beta 0} < t < t_{\beta j_\beta - 1} + \nu T,$$
$$L_{\beta j_\beta}(p) \cdot y_{\beta j_\beta}(t) = \bar{b}_{\beta j_\beta} + \bar{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' M_{\beta\delta}(p)\, y_{\delta j_\beta}(t)$$
$$\text{for} \quad \nu T + t_{\beta j_\beta - 1} < t < t_{\beta j_\beta} + \nu T,$$
$$y_{\beta j_\beta}(t) = 0 \quad \text{for} \quad \nu T + t_{\beta j_\beta} < t < t_{\beta 0} + (\nu + 1)\, T,$$
$$(\nu = -\infty, \ldots, -1, 0, 1, \ldots, +\infty), \tag{11}$$

where $y_{\delta j_\beta}(t)$ is a pulse periodic time function with the period T; this function is equal to $y_\delta(t)$ for $\nu T + t_{\beta j_\beta - 1} < t < t_{\beta j_\beta} + \nu T$, and is equal to zero outside the limits of this interval of the period.

If the solution of Eq. (11) is expressed in the form of a Fourier series, then we obtain

$$y_{\beta j_\beta}(t) = \sum_{r=-\infty}^{\infty} \alpha_{\beta j_\beta r}\, e^{i s_r t} \quad \left(s_r = \frac{2\pi}{T} r\right), \tag{12}$$

where*

$$\alpha_{\beta j_\beta r} = \bar{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' \frac{M_{\beta\delta}(is_r)}{L_{\beta j_\beta}(is_r)} \alpha_{\delta j_\beta r} + \Lambda_{\beta j_\beta r}, \tag{13}$$

and

$$\begin{aligned}\Lambda_{\beta j_\beta r} = \frac{1}{T}\Bigg\{\sum_{k=0}^{n-1} \eta^k_{\beta j_\beta 1} \frac{l_{\beta j_\beta k}(is_r)}{L_{\beta j_\beta}(is_r)} - \Bigg[\bar{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' \sum_{q=0}^{n-2} \eta^q_{\delta j_\beta 1} \frac{m_{\beta\delta q}(is_r)}{L_{\beta j_\beta}(is_r)} - \frac{\bar{b}_{\beta j_\beta}}{is_r L_{\beta j_\beta}(is_r)}\Bigg]\Bigg\} e^{-is_r t_{\beta j_\beta - 1}} \\ + \frac{1}{T}\Bigg\{\sum_{k=0}^{n-1} \eta^k_{\beta j_\beta 2} \frac{l_{\beta j_\beta k}(is_r)}{L_{\beta j_\beta}(is_r)} - \Bigg[\bar{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' \sum_{q=0}^{n-2} \eta^q_{\delta j_\beta 2} \frac{m_{\beta\delta q}(is_r)}{L_{\beta j_\beta}(is_r)} - \frac{\bar{b}_{\beta j_\beta}}{is_r L_{\beta j_\beta}(is_r)}\Bigg]\Bigg\} e^{-is_r t_{\beta j_\beta}},\end{aligned} \tag{14}$$

where

$$l_{\beta j_\beta k}(is_r) = (is_r)^{n-k-1} + \lambda_{\beta j_\beta 1}(is_r)^{n-k-2} + \ldots + \lambda_{\beta j_\beta n-k-1},$$

$$m_{\beta\delta q}(is_r) = \mu_{\beta\delta 1}(is_r)^{n-q-2} + \mu_{\beta\delta 2}(is_r)^{n-q-3} + \ldots + \mu_{\beta\delta n-q-1}.$$

In the formulas (13) and (14) we have adopted the following notation: $\alpha_{\delta j_\beta r}$ is a Fourier coefficient of the periodic pulse function $y_{\delta j_\beta}(t)$;

$\eta^k_{\beta j_\beta 1}$, $\eta^k_{\beta j_\beta 2}$ are the discontinuities in the function $y^{(k)}_{\beta j_\beta}(t)$ for $t = t_{\beta j_\beta - 1}$ and $t = t_{\beta j_\beta}$, respectively;

$\eta^q_{\delta j_\beta 1}$, $\eta^q_{\delta j_\beta 2}$ are the discontinuities in the function $y^{(k)}_{\delta j_\beta}(t)$ for $t = t_{\beta j_\beta - 1}$ and $t = t_{\beta j_\beta}$, respectively.

Equations (13) make it possible to determine the Fourier coefficients $\alpha_{\beta j_\beta r}$, provided only that we know all the Fourier coefficients $\alpha_{\beta j_\beta r}$ ($\delta \neq \beta$). These coefficients can be determined from Eq. (10), but for this purpose it is necessary to specify the sequence of "switching" instants for the various nonlinear characteristics; this appreciably complicates the problem of determining the periodic solutions of the equations of motion (5) in the general case of piecewise-linear characteristics (8), since the number of variants or types of possible periodic solutions increases radically as the number of nonlinearities increases.

Such difficulty does not arise in the case where all the nonlinear characteristics are relay characteristics. In fact, for relay characteristics all angular coefficients are equal to zero:

$$\bar{a}_{\beta j_\beta} = 0 \qquad (\beta = 1, 2, \ldots, m;\ j_\beta = 1, 2, \ldots, N_\beta). \tag{15}$$

In accordance with (13) there is no need to compute the Fourier coefficients $\alpha_{\delta j_\beta}$ when (15) applies; therefore in this case there is no need to specify the switching instants for the various nonlinear characteristics.

We shall study this case in greater detail.

3. Determining Periodic Movements in an Automatic Control Systems with Relay Characteristics

In the case of relay characteristics formulas (13) and (14) yield the following result when (15) is taken into account:

$$\alpha_{\beta j_\beta r} = \frac{1}{T}\sum_{k=0}^{n-1}\Bigg[\eta^k_{\beta j_\beta 1} \frac{n_k(is_r)}{N(is_r)} e^{-is_r t_{\beta j_\beta - 1}} + \eta^k_{\beta j_\beta 2} \frac{n_k(is_r)}{N(is_r)} e^{-is_r t_{\beta j_\beta}}\Bigg] \tag{16}$$

* The derivation of formulas (13) and (14) is performed in section 2 of the Appendix.

$$+\frac{b_{\beta j_\beta}}{is_r T N(is_r)}\left(e^{-is_r t_{\beta j_\beta - 1}} - e^{-is_r t_{\beta j_\beta}}\right),$$

where

$$n_k(is_r) = (is_r)^{n-k-1} + \nu_1 (is_r)^{n-k-2} + \ldots + \nu_{n-k-1}.$$

Substituting (16) into (12), we obtain

$$y_{\beta j_\beta}(t) = \sum_{k=0}^{n-1} [R_k(t - t_{\beta j_\beta - 1})\, \eta^k_{\beta j_\beta 1} + R_k(t - t_{\beta j_\beta})\, \eta^k_{\beta j_\beta 2}]$$
$$+\left[R_n(t - t_{\beta j_\beta - 1}) - R_n(t - t_{\beta j_\beta}) + \frac{1}{\nu_n T}(t_{\beta j_\beta} - t_{\beta j_\beta - 1})\right]\overline{b}_{\beta j_\beta}, \tag{17}$$

where

$$R_k(t) = \frac{1}{T}\sum_{r=-\infty}^{\infty} \frac{n_k(is_r)}{N(is_r)} e^{is_r t}, \quad R_n = \frac{1}{T}\sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{e^{is_r t}}{is_r N(is_r)}. \tag{18}$$

Introducing the matrices

$$\left\| \begin{matrix} y_{\beta j_\beta}(t) \\ y'_{\beta j_\beta}(t) \\ \vdots \\ y^{(n-1)}_{\beta j_\beta}(t) \end{matrix} \right\| = Y_{\beta j_\beta}(t); \quad \left\| \begin{matrix} \eta^0_{\beta j_\beta 1} \\ \eta^1_{\beta j_\beta 1} \\ \vdots \\ \eta^{n-1}_{\beta j_\beta 1} \end{matrix} \right\| = Y_{\beta j_\beta}(t_{\beta j_\beta - 1} + 0) = \Delta Y_{\beta j_\beta 1},$$

$$\left\| \begin{matrix} \eta^0_{\beta j_\beta 2} \\ \eta^1_{\beta j_\beta 2} \\ \vdots \\ \eta^{n-1}_{\beta j_\beta 2} \end{matrix} \right\| = -Y_{\beta j_\beta}(t_{\beta j_\beta} - 0) = \Delta Y_{\beta j_\beta 2};$$

$$\left\| \begin{matrix} R_0(t) & R_1(t) & \ldots R_{n-1}(t) \\ R'_0(t) & R'_1(t) & \ldots R'_{n-1}(t) \\ \ldots & \ldots & \ldots \\ R_0^{(n-1)}(t) & R_1^{(n-1)}(t) & \ldots R_{n-1}^{(n-1)}(t) \end{matrix} \right\| = S(t); \quad \left\| \begin{matrix} R_n(t) \\ R'_n(t) \\ \vdots \\ R_n^{(n-1)}(t) \end{matrix} \right\| = S_n(t);$$

$$\left\| \begin{matrix} \frac{1}{\nu_n T}(t_{\beta j_\beta} - t_{\beta j_\beta - 1}) \\ 0 \\ \vdots \\ 0 \end{matrix} \right\| = T_{\beta j_\beta n}.$$

the solution (17) and its derivatives up to the (n − 1)-th derivative can be written in matrix form

$$Y_{\beta j_\beta}(t) = S(t - t_{\beta j_\beta - 1})\,\Delta Y_{\beta j_\beta 1} + S(t - t_{\beta j_\beta})\,\Delta Y_{\beta j_\beta 2}$$
$$+ [S_n(t - t_{\beta j_\beta - 1}) - S_n(t - t_{\beta j_\beta}) + T_{\beta j_\beta n}]\, b_{\beta j_\beta}. \tag{19}$$

In accordance with the definition of pulse periodic time functions $Y_{\beta j_\beta}(t)$, a periodic solution of Eq. (10) is obtained by the direct summation of these functions. For the case under study we have

$$Y_\beta(t) = \sum_{j_\beta=1}^{N_\beta} \{S(t - t_{\beta j_\beta - 1}) \Delta Y_{\beta j_\beta 1} + S(t - t_{\beta j_\beta}) \Delta Y_{\beta j_\beta 2}$$
$$+ [S_n^{-}(t - t_{\beta j_\beta - 1}) - S_n(t - t_{\beta j_\beta}) + T_{\beta j_\beta n}] \bar{b}_{\beta j_\beta}\} \tag{20}$$
$$(\beta = 1, 2, \ldots, m).$$

The unknowns in (20) are the switching instants $t_{\beta j \beta}$ for all the relay characteristics and elements of the matrix-columns $\Delta Y_{\beta j \beta 1}$, $\Delta Y_{\beta j \beta 2}$. In order to determine them we shall use the conditions governing the transition from one segment of each nonlinear characteristic to the other; in particular, we shall make use of Eq. (9) and the continuity conditions for $Y_\beta(t)$ when $t = t_{\beta j \beta}$.

First we shall take the continuity conditions for $Y_\beta(t)$ which on the basis of (20) yield*

$$Y_\beta(t_{\beta j_\beta} + 0) - Y_\beta(t_{\beta j_\beta} - 0) = \sum_{j'_\beta=1}^{N_\beta} \{[S(t_{\beta j_\beta} + 0 - t_{\beta j'_\beta - 1})$$
$$- S(t_{\beta j_\beta} - 0 - t_{\beta j'_\beta - 1})] \Delta Y_{\beta j'_\beta 1} + [S(t_{\beta j_\beta} + 0 - t_{\beta j'_\beta})$$
$$- S(t_{\beta j_\beta} - 0 - t_{\beta j'_\beta})] \Delta Y_{\beta j'_\beta 2} + [S_n(t_{\beta j_\beta} + 0 - t_{\beta j'_\beta - 1})$$
$$- S_n(t_{\beta j_\beta} - 0 - t_{\beta j'_\beta - 1}) - S_n(t_{\beta j_\beta} + 0 - t_{\beta j'_\beta})$$
$$+ S_n(t_{\beta j_\beta} - 0 - t_{\beta j'_\beta})] \bar{b}_{\beta j_\beta}\}. \tag{21}$$

Taking into account the known properties of the matrices $S_n(t)$, $S(t)$ [3]:

$$S_n(\tau + 0) - S_n(\tau - 0) = 0 \quad \text{for any} \quad \tau,$$
$$S(\tau + 0) - S(\tau - 0) = 0 \quad \text{for} \quad \tau \neq \nu T,$$
$$(\nu = 0, \ \pm 1, \ \pm 2, \ldots),$$
$$S(\nu T + 0) - S(\nu T - 0) = E,$$

where E is a unit matrix, we obtain

$$\Delta Y_{\beta j_\beta + 1, 1} + \Delta Y_{\beta j_\beta 2} = 0$$

or

$$\Delta Y_{\beta j_\beta 2} = - \Delta Y_{\beta j_\beta + 1, 1} \tag{22}$$

from (21).

Substituting (22) into (20) and taking into account the fact that during the periodic movement

$$\Delta Y_{\beta N_\beta + 1, 1} = \Delta Y_{\beta 1, 1},$$

we obtain

$$Y_\beta(t) = \sum_{j_\beta=1}^{N_\beta} [S_n(t - t_{\beta j_\beta - 1}) - S_n(t - t_{\beta j_\beta}) + T_{\beta j_\beta n}] \bar{b}_{\beta j_\beta} \tag{23}$$
$$(\beta = 1, 2, \ldots, m)$$

after appropriate cancellations.

Thus in the relay case the periodic solutions are expressed exclusively in terms of the unknown switching instants $t_{\beta j \beta}$; to determine these instants it is sufficient to require that conditions (9) be satisfied.

*Here j'_β are used to denote those subscripts j_β according to which the summation proceeds in the formula.

Assume $M_{\beta\delta}$ denotes the matrix-row

$$M_{\beta\delta} = \| \mu_{\beta\delta n}\mu_{\beta\delta n-1} \ldots \mu_{\beta\delta 1} \|,$$

where $\mu_{\beta\delta q}$ are the coefficients of the operator $M_{\beta\delta}(p)$ in (4). Then on the basis of (5b) the arguments $\sigma_\beta(t)$ of the relay characteristics are given by formulas

$$\sigma_\beta(t) = \sum_{\delta=1}^{m} M_{\beta\delta} Y_\delta(t) \quad (\beta = 1, 2, \ldots, m). \tag{24}$$

Substituting (24) into (9) and taking (23) into account, we obtain

$$\sum_{\delta=1}^{m} M_{\beta\delta} \sum_{j_\delta=1}^{N_\delta} [S_n(t_{\beta j_\beta} - t_{\delta j_\delta - 1}) - S_n(t_{\beta j_\beta} - t_{\beta j_\delta}) + T_{\delta j_\delta n}] \bar{b}_{\delta j_\delta} = \sigma_{\beta j_\beta}\mathrm{sp} \tag{25}$$
$$(\beta = 1, 2, \ldots, m;\ j_\beta = 1, 2, \ldots, N_\beta).$$

Equations (25), which form a system of $N_1 + N_2 + \ldots + N_m$ equations, are called the equations for the periods. They make it possible to determine all the unknown elements $t_{\beta j_\beta}$ and the period T for the oscillatory movements.

Assume $t_{10} = 0$ is taken as the time origin. Then for $\beta = 1$ there remain N_1 unknowns $t_{11}, t_{12}, \ldots, t_{1N_1}$, where $T = t_{1N_1}$. Taking the relationships

$$t_{\delta 0} = t_{\delta N_\delta} - T \quad (\delta = 2, 3, \ldots, m)$$

into account as well, we obtain N_β unknowns $t_{\beta 1}, t_{\beta 2}, \ldots, t_{\beta N\beta}$ $(\beta = 1, 2, \ldots, m)$ each for all the nonlinearities; i.e., we obtain the same number of unknowns as the number of equations in the system (25).

Each solution of the equations for the periods, (25), taken together with (23), (24), actually determines a periodic movement if the following conditions are satisfied:

a) The switching conditions $t_{\beta j_\beta}$ for all the nonlinear characteristics as obtained from the equations for the periods, (25), satisfy conditions (7);

b) there are no other relay switchings except those which occur at the instants $t_{\beta j_\beta}$; these switchings are directed in such a way as to assure that for $t = t_{\beta j_\beta - 1} + 0$ the image point $\sigma_\beta, f_\beta(\sigma_\beta)$ begins movement over the segment with the ordinate $\bar{b}_{\beta j_\beta}$, and then leaves this segment and begins movement over the next segment at $t = t_{\beta j_\beta} - 0$.

Mathematically, condition (b) means that if we formulate the solution $\sigma_\beta(t)$ from formulas (24) while taking (23) into account for the elements $t_{\beta j_\beta}$ obtained from the equations of the periods, (25), it follows that for all β the relationships

$$\begin{aligned}
&\sigma_\beta(t) \neq \sigma_{\beta j_\beta - 1}\,\mathrm{sp},\ \sigma_{\beta j_\beta}\,\mathrm{sp} && \text{for} \quad t_{\beta j_\beta - 1} < t < t_{\beta j_\beta},\\
&\sigma_\beta(t) = \sigma_{\beta j_\beta - 1}\,\mathrm{sp},\ \sigma_{\beta j_\beta}\,\mathrm{sp} && \text{for} \quad t = t_{\beta j_\beta - 1},\ t_{\beta j_\beta},\\
&\dot{\sigma}_\beta(t_{\beta j_\beta - 1} + 0)\,\dot{\sigma}_\beta(t_{\beta j_\beta} + 0) > 0 && \text{for} \quad \sigma_{\beta j_\beta - 1}\,\mathrm{sp} \neq \sigma_{\beta j_\beta}\,\mathrm{sp},\\
&\dot{\sigma}_\beta(t_{\beta j_\beta - 1} + 0)\,\sigma_\beta(t_{\beta j_\beta} + 0) < 0 && \text{for} \quad \sigma_{\beta j_\beta - 1}\,\mathrm{sp} = \sigma_{\beta j_\beta}\,\mathrm{sp}
\end{aligned}$$

apply.

The solutions of Eq. (25) which do not completely satisfy at least one of the conditions "a" and "b" must be discarded, since they cannot determine periodic movements of the specified type.

From the coordinates $Y_\beta(t)$ of the periodic mode (23) it is easy to make the transition to the original coordinates $x_k(t)$. Thus if M_k^δ is the matrix-row

$$M_k^\delta = \| \mu_{kn}^\delta\ \mu_{kn-1}^\delta \ldots \mu_{k1}^\delta \|,$$

where μ^{δ}_{kl} are the coefficients of the operator $M^{\delta}_{k}(p)$ from (3), it follows that on the basis of (6) we have

$$x_k(t) = \sum_{\delta=1}^{m} M^{\delta}_{k} Y_{\delta}(t).$$

The case where all the relay characteristics are symmetrical is of independent interest (i.e., the case where we have

$$f_{\beta}[\sigma_{\beta}(t)] = -f_{\beta}\left[\sigma_{\beta}\left(t+\frac{T}{2}\right)\right]$$

for all the nonlinearities when the movement is periodic).

Under these conditions the number of unknowns $t_{\beta j_{\beta}}$ and the number of equations for the periods,(25), are reduced by a factor of 2: $j_{\beta} = 1, 2, \ldots, N_{\beta}/2$ for all β.

The method developed here for determining the period movements can also be used in finding forced oscillations of a relay system with m relay elements and arbitrary linear sections. In that case the time origin t_{10} which is due to period constraining inputs is determined along with the remaining unknowns; the period of the forced oscillations is known: it is equal to the period T of the constraining inputs for the "fundamental oscillations," and is equal to νT (ν is an integer) for the "subharmonic oscillations."

The elements of the matrix-column $S_n(t)$ that appear in formula (23) and the equations for the periods,(25), are derived from the corresponding formula (18) in the form of complete Fourier series. In that form they are determined exclusively by the coefficients $\nu_1, \nu_2, \ldots, \nu_n$ of the operator (2) which are expressed in terms of the parameters of the original system.

From complete Fourier series it is possible to make the transition to the closed form of the elements of the matrix-column $S_n(t)$ which is expressed in terms of the roots of the characteristic polynomial N(p).

In the Appendix, section 3, two methods are given for computing the function $R_n(t)$ and its derivatives: a) an approximate method based on summing the corresponding Fourier series, b) and an exact method based on integrating a linear differential equation of the n-th order with a right side that is satisfied by $R_n(t)$.

Appendix

1. Proof of Equivalence for Eq. (1) and (5)

We shall write Eq. (1a) in the following form:

$$\sum_{\beta=1}^{m} b_{k\beta} f_{\beta}(\sigma_{\beta}) = \dot{x}_k - \sum_{\alpha=1}^{n} a_{k\alpha} x_{\alpha} \quad (k = 1, 2, \ldots, n). \tag{26}$$

Equations (5a) can be reduced to the same form. Under these conditions we shall study the known relationship between the determinant N(p) and its algebraic complements [4]:

$$\delta_{sk} N(p) \equiv p N_{sk}(p) - \sum_{\alpha=1}^{n} a_{k\alpha} N_{s\alpha}(p). \tag{27}$$

Multiplying the left and right sides of the identity (27) by $b_{s\beta}$ and summing over all $s = 1, 2, \ldots, n$ while taking (3) into account, we obtain

$$b_{k\beta} N(p) \equiv p M^{\beta}_{k}(p) - \sum_{\alpha=1}^{n} a_{k\alpha} M^{\beta}_{\alpha}(p). \tag{28}$$

We now take Eq. (5a) and multiply its left and right sides by $b_{k\beta}$:

$$b_{k\beta} f_{\beta}(\sigma_{\beta}) = b_{k\beta} N(p)\, y_{\beta}. \tag{29}$$

Substituting (28) into (29) and summing over all $\beta = 1, 2, \ldots, m$ in the left and right sides, we obtain

$$\sum_{\beta=1}^{m} b_{k\beta} f_\beta(\sigma_\beta) = p \sum_{\beta=1}^{m} M_k^\beta(p)\, y_\beta - \sum_{\alpha=1}^{n} a_{k\alpha} \sum_{\beta=1}^{m} M_\alpha^\beta(p)\, y_\beta$$

or, taking (6) into account,

$$\sum_{\beta=1}^{m} b_{k\beta} f_\beta(\sigma_\beta) = \dot{x}_k - \sum_{\alpha=1}^{n} a_{k\alpha} x_\alpha \tag{30}$$

If in (30) we assign all values from 1 to $\underline{n}$ to the subscript $\underline{k}$, we again obtain Eq. (26). By the same token the equivalence of Eq. (1a) and (5a) is proven.

We shall now prove that Eq. (1b) and (5b) are equivalent. For this purpose we take any Eq. (1b) and substitute (6) into it:

$$\sigma_\beta = \sum_{\gamma=1}^{n} c_{\beta\gamma} x_\gamma = \sum_{\gamma=1}^{n} c_{\beta\gamma} \sum_{\delta=1}^{m} M_\gamma^\delta(p)\, y_\delta. \tag{31}$$

Changing the summation order in (31) and taking (4) into account we obtain the corresponding Eq. (5b).

2. Derivation of Formulas (13) and (14)

If D is the symbol for the generalized time derivative [5], then the relationships

$$\begin{aligned} L_{\beta j_\beta}(D)\, y_{\beta j_\beta}(t) = L_{\beta j_\beta}(p)\, y_{\beta j_\beta}(t) + \eta^0_{\beta j_\beta 1} \sum_{\nu=-\infty}^{\infty} [\delta^{(n-1)}(t - t_{\beta j_\beta - 1} - \nu T) \\ + \lambda_{\beta j_\beta 1} \delta^{(n-2)}(t - t_{\beta j_\beta - 1} - \nu T) + \ldots + \lambda_{\beta j_\beta n-1} \delta(t - t_{\beta j_\beta - 1} - \nu T)] \\ + \eta^0_{\beta j_\beta 2} \sum_{\nu=-\infty}^{\infty} [\delta^{(n-1)}(t - t_{\beta j_\beta} - \nu T) + \lambda_{\beta j_\beta 1} \delta^{(n-2)}(t - t_{\beta j_\beta} - \nu T) + \ldots \\ \ldots + \lambda_{\beta j_\beta n-1} \delta(t - t_{\beta j_\beta} - \nu T)] + \ldots + \eta^{n-1}_{\beta j_\beta 1} \sum_{\nu=-\infty}^{\infty} \delta(t - t_{\beta j_\beta - 1} - \nu T) \\ + \eta^{n-1}_{\beta j_\beta 2} \sum_{\nu=-\infty}^{\infty} \delta(t - t_{\beta j_\beta} - \nu T), \end{aligned} \tag{32}$$

$$\begin{aligned} M_{\beta\delta}(D)\, y_{\delta j_\beta}(t) = M_{\beta\delta}(p)\, y_{\delta j_\beta}(t) + \eta^0_{\delta j_\beta 1} \sum_{\nu=-\infty}^{\infty} [\mu_{\beta\delta 1} \delta^{(n-2)}(t - t_{\beta j_\beta - 1} - \nu T) + \ldots \\ + \mu_{\beta\delta n-1} \delta(t - t_{\beta j_\beta - 1} - \nu T)] + \eta^0_{\delta j_\beta 2} \sum_{\nu=-\infty}^{\infty} [\mu_{\beta\delta 1} \delta^{(n-2)}(t - t_{\beta j_\beta} - \nu T) + \ldots \\ + \mu_{\beta\delta n-1} \delta(t - t_{\beta j_\beta} - \nu T)] + \ldots + \eta^{n-1}_{\delta j_\beta 1} \sum_{\nu=-\infty}^{\infty} \mu_{\beta\delta 1} \delta(t - t_{\beta j_\beta - 1} - \nu T) \\ + \eta^{n-1}_{\delta j_\beta 2} \sum_{\nu=-\infty}^{\infty} \mu_{\beta\delta 1} \delta(t - t_{\beta j_\beta} - \nu T) \end{aligned} \tag{33}$$

apply, where $\delta(t)$ is the Dirac pulse function.

On the other hand, we also have

$$L_{\beta j_\beta}(D)\, y_{\beta j_\beta}(t) - \sum_{r=-\infty}^{\infty} L_{\beta j_\beta}(is_r)\, \alpha_{\beta j_\beta r} e^{is_r t}, \tag{34}$$

$$M_{\beta\delta}(D)\, y_{\delta j_\beta}(t) = \sum_{r=-\infty}^{\infty} M_{\beta\delta}(is_r)\, \alpha_{\delta j_\beta} e^{is_r t}. \tag{35}$$

The pulse periodic function $\sum_{\nu=-\infty}^{\infty} \delta^{(q)}(t-\nu T)$ can also be expanded into a Fourier series:

$$\sum_{\nu=-\infty}^{\infty} \delta^{(q)}(t-\nu T) = \frac{1}{T} \sum_{r=-\infty}^{\infty} (is_r)^q e^{is_r t} \tag{36}$$

From (33) we obtain

$$M_{\beta\delta}(p)\, y_{\delta j_\beta}(t) = \sum_{r=-\infty}^{\infty} \Big\{ M_{\beta\delta}(is_r)\, \alpha_{\delta j_\beta r} - \frac{1}{T} \sum_{q=0}^{n-2} \Big[\eta^q_{\delta j_\beta 1} m_{\beta\delta 1}(is_r)\, e^{-is_r t_{\beta j_\beta - 1}} + \eta^q_{\delta j_\beta 2} m_{\beta\delta q}(is_r)\, e^{-is_r t_{\beta j_\beta}} \Big] \Big\} e^{is_r t}. \tag{37}$$

while taking (35) and (36) into account.

We now express the pulse periodic function, which consists of square pulses with a height $\overline{b}_{\beta j_\beta}$ and a width $t_{\beta j_\beta} - t_{\beta j_\beta - 1}$ and is included in the second of Eqs. (11), in the form of a Fourier series:

$$\overline{b}_{\beta j_\beta} = \overline{b}_{\beta j_\beta} \frac{1}{T} \sum_{r=-\infty}^{\infty} \frac{e^{-is_r t_{\beta j_\beta - 1}} - e^{-is_r t_{\beta j_\beta}}}{is_r} e^{is_r t}. \tag{38}$$

If we introduce (37) and (38) into the second of Eqs. (11), then we obtain

$$\begin{aligned} L_{\beta j_\beta}(p)\, y_{\beta j_\beta}(t) = \frac{1}{T} \sum_{r=-\infty}^{\infty} \Big\{ & T \overline{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' M_{\beta\delta}(is_r)\, \alpha_{\delta j_\beta r} \\ & - \Big[\overline{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' \sum_{q=0}^{n-2} \eta^q_{\delta j_\beta 1} m_{\beta\delta q}(is_r) - \frac{1}{is_r} \overline{b}_{\beta j_\beta} \Big] e^{-is_r t_{\beta j_\beta - 1}} \\ & - \Big[\overline{a}_{\beta j_\beta} \sum_{\delta=1}^{m}{}' \sum_{q=0}^{n-2} \eta^q_{\delta j_\beta 2} m_{\beta\delta q}(is_r) - \frac{1}{is_r} \overline{b}_{\beta j_\beta} \Big] e^{-is_r t_{\beta j_\beta}} \Big\} e^{is_r t}. \end{aligned} \tag{39}$$

Substituting (34), (36) and (39) into (32) and setting the coefficients of identical harmonics equal to each other, we obtain formulas (13) and (14).

3. The Methods for Computing the Function $R_n(t)$ and Its Derivatives

a) Approximate method. Making use of the relationship

$$\frac{1}{sN(s)} \equiv \sum_{i=1}^{l} \frac{d_i}{s^{n+1}} + \frac{e_1 s^{n-1} + e_2 s^{n-2} + \ldots + e_n}{s^{n+l} N(s)} \qquad (s \neq 0),$$

where d_i, e_k are coefficients which are determined from the equations

$$\begin{gathered} d_1 = 1, \\ d_2 + \nu_1 d_1 = 0, \\ \cdots\cdots \\ d_l + \nu_1 d_{l-1} + \ldots + \nu_{l-1} d_1 = 0, \\ \nu_1 d_l + \nu_2 d_{l-1} + \ldots + \nu_l d_1 + e_1 = 0, \end{gathered}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$v_{n-1}d_l + v_n d_{l-1} + e_{n-1} = 0,$$

$$v_n d_l + e_n = 0,$$

the function $R_n(t)$ from (18) is represented in the form

$$R_n(t) = \sum_{i=1}^{l} d_i \rho_{n+i}(t) + R_{n_*}(t), \tag{40}$$

where

$$\rho_q(t) = \frac{1}{T} \sum_{r=-\infty}^{\infty} \frac{e^{is_r t}}{(is_r)^q}, \tag{41}$$

$$R_{n_*}(t) = \frac{1}{T} \sum_{\substack{r=-\infty \\ r \neq 0}}^{\infty} \frac{e_1 (is_r)^{n-1} + e_2 (is_r)^{n-2} + \ldots + e_n}{(is_r)^{n+l} N(is_r)} e^{is_r t}. \tag{42}$$

The derivatives $R_n^{(k)}(t)$ $(k = 1, 2, \ldots, n-1)$ can be obtained from a term-by-term differentiation of (40):

$$R_n(t) = \sum_{i=1}^{l} d_i \rho_{n+i-k}(t) + R_{n_*}^{(k)}(t). \tag{43}$$

The functions $\rho_q(t)$ which appear in (40), (43) and are determined from (41) are computed exactly [3], and the function $R_{n_*}(t)$ and its derivatives are rapidly converging Fourier series which can be computed by limiting ourselves to the first harmonic. Under these conditions the integer l is determined while neglecting the higher harmonics on the basis of the required accuracy with which the series must be computed.

<u>b) The exact method.</u> If we apply the operator N(D) to the function $R_n(t)$ from (18), we obtain

$$N(D) R_n(t) = \frac{1}{T} \sum_{\substack{r=-\infty \\ r \neq 0}}^{\infty} \frac{e^{is_r t}}{is_r} = \rho_1(t), \tag{44}$$

where $\rho_1(t)$ is a known sawtooth function with the period T. In view of the continuity of $R_n(t)$ and its derivatives up to the (n−1)-th derivative inclusive, the operator for the generalized derivative D in (44) can be replaced by the ordinary differentiation operator <u>p</u>:

$$N(p) R_n(t) = \rho_1(t). \tag{45}$$

Integrating Eq. (45) in the interval $0 < t < T$, we obtain

$$S_n(t) = U(t) C + V(t), \tag{46}$$

where

$$U(t) = \left\| \begin{matrix} e^{p_1 t} & e^{p_2 t} & \ldots & e^{p_n t} \\ p_1 e^{p_1 t} & p_2 e^{p_2 t} & \ldots & p_n e^{p_n t} \\ \cdots & \cdots & \cdots & \cdots \\ p_1^{n-1} e^{p_1 t} & p_2^{n-1} e^{p_2 t} & \ldots & p_n^{n-1} e^{p_n t} \end{matrix} \right\|, \quad C = \left\| \begin{matrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{matrix} \right\|.$$

$$V(t) = \left\| \begin{matrix} -\dfrac{t}{\nu_n T} + \dfrac{1}{2\nu_n} + \dfrac{\nu_{n-1}}{\nu_n T} \\ -\dfrac{1}{\nu_n T} \\ 0 \\ \vdots \\ 0 \end{matrix} \right\| .$$

Here p_k are the roots of the characteristic equation $N_{(p)} = 0$, and are assumed to be constant.

The integration constants C_k are determined from the condition governing the periodicity $S_n(t)$; this condition can be written as

$$S_n(0) - S_n(T) = 0. \tag{47}$$

Substituting (46) into (47), we obtain

$$C = [U(0) - U(T)]^{-1} B, \tag{48}$$

where

$$B = V(T) - V(0) = \left\| \begin{matrix} -\dfrac{1}{\nu_n} \\ 0 \\ \vdots \\ 0 \end{matrix} \right\| . \tag{49}$$

After certain algebraic transformations the elements of the column (48) are reduced to the form

$$C_k = \frac{(-1)^{n+k}}{\left(1 - e^{p_k T}\right) p_k} \left[\frac{1}{p_k - p_{k-1}} + \frac{1}{p_k - p_{k+1}} \right] \tag{50}$$
$$(k = 1, 2, \ldots, n;\ p_0 = p_n;\ p_{n+1} = p_1)$$

when (49) is taken into account.

On the basis of (50) all the elements of the matrix-column $S_n(t)$ [i.e., the function $R_n(t)$ and its first $n-1$ derivatives] can be determined exactly in accordance with (46) using the roots of the characteristic equation for the system.

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ON THE PROBLEM OF DESIGNING HIGH-SPEED AUTOMATIC CONTROLLERS FOR INDUSTRIAL OBJECTS

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A block diagram is given for a controller which reproduces an almost optimal control input under conditions where the coordinate of the control organ is bounded; the results of an investigation of the dynamic properties of the controller are also cited.

The basic problem which arises in designing an automatic controller for industrial objects is the assurance of its maximum efficiency of operation. This problem gives rise to two basic requirements. On the one hand, the efficiency of the controller operation is determined by the degree to which the law governing the variation of the controlled quantity corresponds to the specified law. It is from this that the first requirement derives – an increase in dynamic accuracy. On the other hand the efficiency of controller operation is determined by its reliability (i.e., by the efficiency of its elements, etc.). This is the basis for the second requirement – simplification of the controller and an increase in the reliability of its elements.

The degree to which each of these requirements is satisfied depends on the specific conditions governing the use of the controller and is determined to a considerable extent by the special features involved in the technology of the automatized process. Therefore in the general case it is not possible to indicate which of these requirements is more important.

It is possible, however, to state that for objects with unfavorable dynamic characteristics (for example, objects with a large ratio between the lag and the time constant) the achievement of a high dynamic accuracy is associated with great difficulties. It is in this sense that the design of controllers which assure a higher dynamic accuracy is of definite practical interest.

The Block Diagram of the Controller

We shall study an automatic control system whose block diagram is shown in Fig. 1. The transfer function for the object is written as

$$W(p) = \frac{ke^{-\tau p}}{(T_1 p + 1)(T_2 p + 1)}, \tag{1}$$

where k is the transfer coefficient, T_1 and T_2 are the time constants, τ is the constant lag.

Expression (1) describes the dynamic properties of a rather wide class of industrial automatic control objects.

Assume that based on the technological conditions a sufficiently rapid displacement of the controlling organ is allowed. In that case the servomotor time can be assumed sufficiently small compared to the time constant of the object.

Assume further that the engineering-economic characteristics of the examined class of objects are such that the requirements imposed on the accuracy with which the controlled quantity is maintained are dictated not only by considerations of safe operation of the equipment, but also by economic considerations. In other words, as the dynamic error of an automatic control system increases, the economic indices for the operation of the equipment deteriorate. Then for the class of objects under study the problem reduces to designing a controller which is optimal according to a definite criterion.

The theoretical bases for designing systems that are optimal in the sense of transient response duration are cited in [1-3] for typical perturbations.

When the shape of the perturbations is complex (this is characteristic for industrial objects) the deviation of the controlled quantity from the specified value plays an essential part. From the point of view of the technician it is the most important parameter according to which he can make a judgement concerning the dynamic properties of the controller used on the object.

Assume that the relationship between the deviation $\underline{x}$ of the controlled quantity and the generalized loss L is that shown in Fig. 2; assume also that there exists a region of allowed states of the object confined within the limits

$$x_1 \leqslant x \leqslant x_2. \tag{2}$$

It is obvious that the optimum mode of operation for the object would be the mode for which

$$x = x_2. \tag{3}$$

However, condition (3) is not realizable from the practical point of view, since x(t) deviates from the specified value due to the effect of perturbations. Therefore it is necessary to choose the specified value of the controlled quantity so as to avoid deviations of x(t) beyond an allowable limit.

Thus the quality of control achieved by the controller is defined uniquely by the shaded area (Fig. 2).

Therefore the criterion under study can be written in the form

$$I[x(t)] = \lim_{T\to\infty} \frac{l}{T} \int_0^T [x(t)_{\max} - x(t)]\, dt, \tag{4}$$

where l is a coefficient that characterizes the variation of the generalized loss when the controlled quantity changes by one unit, $x(t)_{\max}$ is the maximum deviation of the controlled quantity, T is the integration period.

From Fig. 2 it follows that the criterion (4) is satisfied in the optimal manner by a controller such that $x = x_2$; this corresponds to the optimum mode of operation for the object. In that case $x(t)_{\max} - x(t) = 0$. Therefore an increase in the dynamic accuracy of the automatic control system is associated with the minimization of the functional (4).

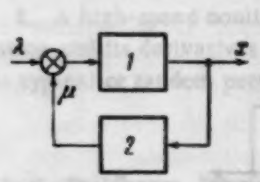

Fig. 1. Block diagram of the automatic control system. 1) Controlled object; 2) controller; λ) perturbation; μ) control signal; x) deviation of the controlled quantity.

One of the effective methods for minimizing this functional is the use of a controller with optimal speed of response. Here it is important to assure a transient response with a minimal deviation of the controlled quantity. In duration the transient response may differ appreciably from the optimal duration. This makes it possible to achieve a substantial simplification of the controller block diagram by avoiding the use of a computer device in it.

Figure 3 shows the block diagram of a controller which achieves almost optimal control. When a signal $\underline{x}$ appears at the input to the controller the relay 6 is energized through the amplifier 5; the relay in turn triggers the pulse generator 7 and the time-specifying unit 8 that consists of a controlled time relay. A time-specifying unit switches on the servomotor 11 by means of the relay 9, and the servomotor displaces the controlling organ to one of the limiting positions (depending on the sign of the signal).

The time t_1 during which the controlling organ is in the limiting position is determined by the sum of the error signal and its first and second derivatives; this sum is applied to the amplifier 4. The position of the controlling organ after the pulse t_1 has been responded to is determined by the magnitude of the signal at the output of the integrating section which is connected for a time t_1 by means of the switch K. The signal corresponding to the position of the controlling organ is applied to the coincidence network 12 and is compared with the signal at the output of section 10. The coincidence network is designed in such a way that when its input is subjected to signals that differ in absolute magnitude its output produces a signal that permits the operation of a time-specifying unit. When the signals at the input of the coincidence network are equal the signal at its output is equal to zero. In that

case the servomotor is switched off, since the time-specifying unit ceases to operate. The repeated connection of the time-specifying unit is achieved by the pulse generator 7 which operates until the signal $\underline{x}$ becomes equal to zero.

Thus the controller under study is a nonlinear controller of the discrete type in its principle of operation; its speed of response is assured by the rapid displacement of the controlling organ to one of the limiting positions, followed by a stay in that position during a definite time that corresponds to the deviation of the controlled quantity and its first and second derivatives, followed finally by the return of the servomotor to the position which approximately corresponds to the equilibrium state of the automatic control system.

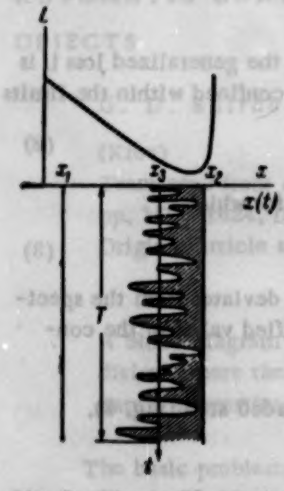

Fig. 2. Figure illustrating the determination of the criterion for the dynamic accuracy of the automatic control system.

The Dynamic Properties of a System that Incorporates a Controller with a High Speed of Response

Figure 4 shows graphs of the transient responses for step perturbations; these graphs were obtained by simulating an object with a fast-acting controller of the type described above, and with a linear "PID" controller. The latter was an electronic controller of the type "ĖRIII-K" that is used for the automatization of thermal processes in electric power stations and has an electronic differentiator at its input. The parameters of the settings for the controllers were chosen to be optimal and were not altered when the magnitude of the perturbations varied.

Figure 5 shows graphs of the controlled processes for random perturbations. The graph of the perturbations was obtained using a real object and represents the time variation of a boiler aggregate load.

In accordance with the criterion cited above a high-speed nonlinear controller assures a control performance that is 1.87 times better than the control performance achieved with a linear "PID" controller.

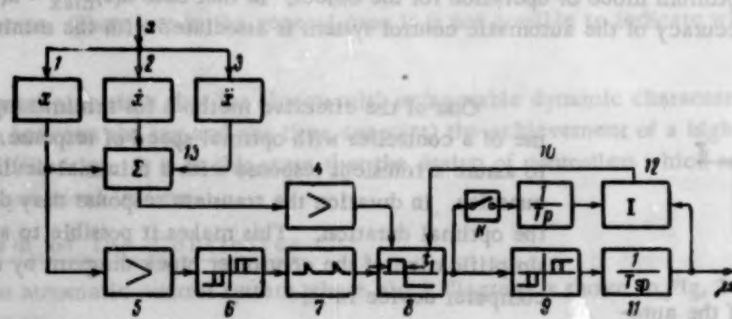

Fig. 3. Block diagram of a fast-acting controller. 1, 2, 3, 13) detecting block; K) switch which closes for a time t_1.

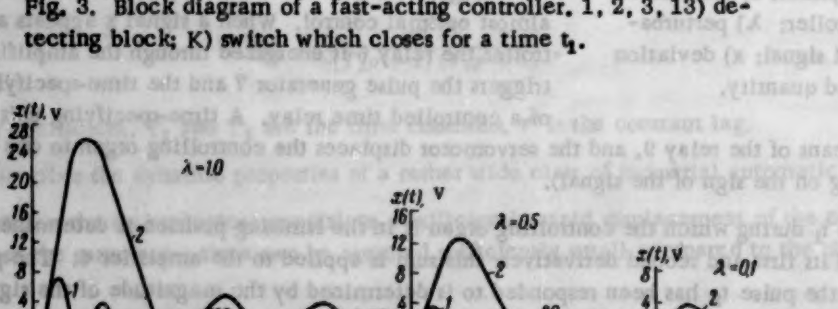

Fig. 4. Graphs of the transient responses. 1) Automatic control system with a fast-acting controller; 2) automatic control system with a "PID" controller. The parameters of the object were: $T_1 = T_2 = 20$ sec, $\tau = 10$ sec.

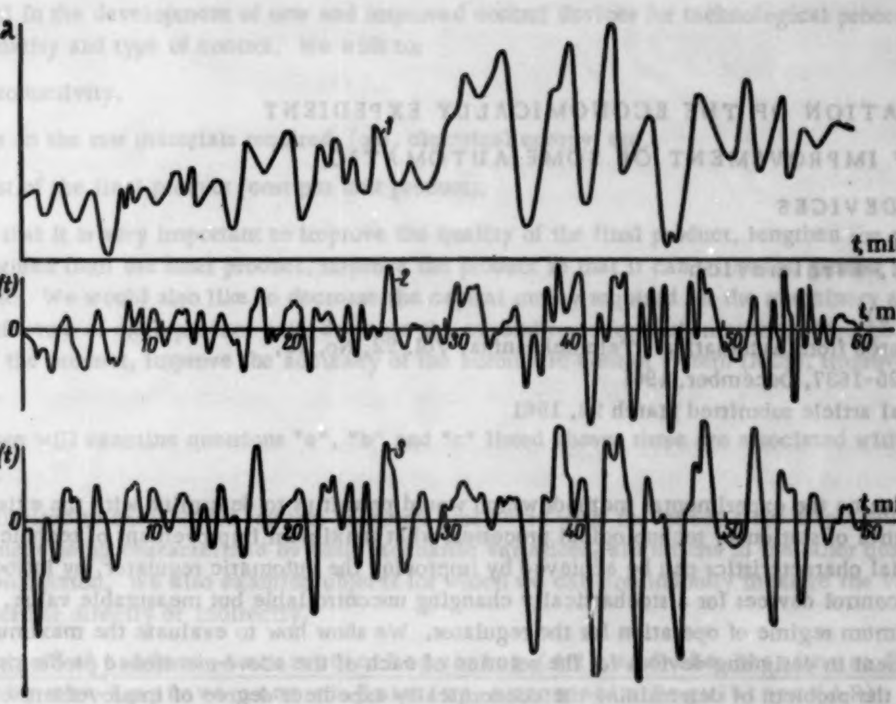

Fig. 5. Graphs for the variation of the controlled variable for a random perturbation. 1) Perturbation (load variation); 2) automatic control system with a high-speed contoller; 3) automatic control system with a "PID" controller. Object parameters: $T_1 = T_2 = 20$ sec, $\tau = 10$ sec.

CONCLUSIONS

1. The use of a performance criterion that takes into account the relationship between the dynamic accuracy of the controller and its economic efficiency proves the basis for designing a controller which a) is almost optimal with respect to speed of response, and b) has a simplified block diagram.

2. A high-speed nonlinear controller which forms the control signal from the sum of the controlled-quantity deviation and its derivatives assures an appreciably better control performance than a linear "PID" controller for either typical or random perturbations.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

DETERMINATION OF THE ECONOMICALLY EXPEDIENT DEGREE OF IMPROVEMENT OF SOME AUTOMATIC CONTROL DEVICES

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We examine the experimental methods which would permit us to determine, with the existing devices for control of stationary technological processes, what maximum improvement of technical-economic industrial characteristics can be achieved by improving the automatic regulator, by introducing automatic control devices for a stochastically changing uncontrollable but measurable value, by choosing an optimum regime of operation for the regulator. We show how to evaluate the maximum capital investment in designing devices for the solution of each of the above-mentioned problems. A solution to the problem of determining the economically expedient degree of improvement of a control device is formulated.

Formulation of the Problem

At present we evaluate the technical-economic effectiveness (TEE) of new or improved automatic control devices by comparing the operating characteristics of the aggregate for the existing and the new devices. Usually this evaluation is performed after the device has been developed for use with the industrial aggregate; we then often discover that the additional effectiveness of using new and highly perfected devices is insignificant while the period of time required to amortize the capital investment is greater than the allowed time.

Under the existing methods of technical process control many physical quantities are not automatically contolled. Naturally their values, as a rule, do not satisfy the conditions required for obtaining optimum technical-economic production characteristics (TEC). In connection with this we should determine for every nonregulated magnitude the maximum improvement in the TEC of production associated with the creation of automatic control devices for this magnitude and also with the optimization of the mode of operation of the proposed automatic regulator (AR). It is also useful to determine the economically feasible degree of capitalization associated with the creation of the new control device.

It is desirable to obtain the above information prior to the organization of the work on automatic control. This may be done by using the existing methods for studying the control of technological processes.

The uninterrupted progress in the theory and methods of automatic regulation and control necessitates a periodic (after five-ten years) examination of the expediency of modernizing or even replacing the existing automatic control devices with more perfected controls.

An important practical role will be played in these decisions by methods which permit us to determine the maximum improvement in the TEC of production associated with the absolute perfection of control devices for each physical magnitude associated with a process.

We must also determine beforehand the improvement in the TEC of production resulting from each improvement in the circuit or construction of the automatic control device, and the economic justification for the improvement of each of these devices.

In this article we will examine several approximate solutions for the problems which we have outlined above.

The final goal in the development of new and improved control devices for technological process is the improvement in the quality and type of control. We wish to:

a) improve productivity,

b) economize on the raw materials required, fuel, electrical energy, etc.,

c) cut the cost of the final product (cost per unit product).

We also find that it is very important to improve the quality of the final product, lengthen the period of service of articles manufactured from the final product, improve the product so that it can withstand greater loading and more rugged use, etc. We would also like to decrease the capital outlay required for the machinery and plant involved in the manufacture of each product unit, decrease the expenditures for fuel, material, etc., per mean unit period of service of the product, improve the accuracy of the automatic control system (ACS), improve the productivity of labor, etc.

In this study we will examine questions "a", "b", and "c" listed above; these are associated with the improvement of some ACS.

Our study is limited to cases where random processes associated with the steady state affect only one physical quantity, these changes being characterized by basic stochastic variations; alterations in the other quantities do not affect the TEC of production. We also examine objects for which we can continuously measure the values of the TEC of production either directly or indirectly.

1. The Concept of the Ideal Automatic Regulator (AR) and the Maximum Effectiveness Obtained by the Improvement of Existing Automatic Regulators (AR)

In order to solve the problems which we have outlined above we must first define the ideal automatic regulator (AR).

We will consider an ideal automatic regulator to be one which, when perturbations of any amplitude act upon the regulated object, instantaneously works out the required regulating action. It is obvious that with an ideal AR the regulated value of the magnitude will always coincide with the given or desired value.

Let us assume that with an ideal AR the productivity, the expenditures for fuel and electrical energy and the cost per unit product are given by Π_{id}, W_{id} and C_{id}. Let us use the letters Π, W and C to denote the corresponding quantities for existing types of AR.

If we consider Π_{id}, W_{id} and C_{id} to be the limiting quantities, that is the values in the ideal case, then we can compute the maximum TEE of the existing AR by determining the following:

$$\delta\Pi = \Pi_{id} - \Pi, \quad -\delta W = W_{id} - W, \quad -\delta C = C_{id} - C. \tag{1}$$

The introduction of the concept of the ideal AR has the same meaning as for example the introduction of the concept of an ideal efficiency of 100% to be used as a standard in evaluating the efficiency of actual machines and aggregates.

2. Technical-Economic Indices to be Used in Evaluating the Operation of Ideal and Actual Automatic Regulators (AR)

Let us assume that the given or desired value of the stochastically changing controlled magnitude

$$x = x_q \tag{2}$$

assures the optimum required complex TEC of production $\Phi(x)$. $\Phi(x)$ can be, for example, the minimum cost C of the unit product. Thus we can find the value of Φ either directly as a function of $\underline{x}$ or by means of other magnitudes $F_i(x)$ which in turn are functions of $\underline{x}$ and which characterize the course of the process and its TEC. F_i may for example be the value of the productivity Π, the efficiency η, the input energy P, etc. Obviously the optimum values of the index Φ (minimum value of C) and of the F_i (i.e., Π, η, P,...) may differ from their maximum values $F_{i\,max}$, i.e., from Π_{max}, η_{max}, P_{max}, etc.

In the control process the values of $\underline{x}$ deviate from x_q; these deviations are compensated for by AR. If for the process under control we can obtain the probability distribution W(x) experimentally, then we can calculate the

mathematical expectation of the regulated magnitude

$$M[x] = \int_{-\infty}^{\infty} xw(x)\,dx \tag{3}$$

and the root-mean-square

$$\bar{x}_{ms} = \sqrt{\int_{-\infty}^{\infty} x^2 w(x)\,dx}\,. \tag{4}$$

Therefore $\bar{x}_{ms} > M[x]$.

Variations in $\underline{x}$ result in changes in the values of the functions $F_i(x)$ and in the indices $\Phi(x)$.

If the statistical characteristic $F(x)$ is a linear function (line 1, Fig. 1) then to the value $x = M[x]$ there corresponds the function $F(x) = F^*_{id}$. In this case as $\underline{x}$ deviates from $M[x]$ for any probability distribution, $w(x)$ is given by

$$M[F] = \int_{0}^{\infty} F(x)\,w(x)\,dx = F'_{id}. \tag{5}$$

The mathematical expectation will coincide with the value $F(x) = F^*_{id}$.

If the statistical characteristic $F(x)$ is a nonlinear function of $\underline{x}$ (curve 2, Fig. 1) then the value $x = M[x]$ will correspond to the value of the function $F(x) = F'_{id}$. In the control process when $x_q = M[x]$ then in the general case,* the mathematical expectation will be given by

$$M[F] = \int_{0}^{\infty} F(x)\,w(x)\,dx < F'_{id}. \tag{6}$$

The greater the curvature of the characteristic $F(x)$ in the vicinity of $M[x]$ and the larger the dispersion $D[F]$, the larger the differential

$$\delta F'_{id} = F'_{id} - M[F]. \tag{7}$$

Of the three distribution functions for $\underline{x}$ shown dashed in Fig. 1, we will obtain the smallest deviation $\delta F'_{id}$ if we use a distribution approximating the normal distribution (curve 3); the largest deviation will be obtained from the distribution characterized by curve 5.

Therefore we should examine the following associated values of the indices $F(x)$ and regulated magnitudes $\underline{x}$ (Fig. 2): x_g and the value F_{id} which is associated with x_g on the statistical characteristic; the mathematical expectation $M[x]$ for a stochastic perturbation** and the corresponding value F'_{id}; the mean value $M[F]$ occurring in the control process and the associated statistical characteristic x_F; the optimum value $F_0^{(x)}$ and the asociated value x_0, etc.

Thus in the control process for a given or desired value x_g the ideal AR would ensure that we obtain $F(x) = F_{id}$ while in the most general case the real AR would, under these conditions, assure a value $M[F] < F_{id}$.

If $F(x)$ is a linear function then, as a result of the product properties or of the type or course of the process, limitations are imposed upon $\underline{x}$ and $F(x)$. The dispersion of the values of $\underline{x}$ or $F(x)$ determines the choice of x_g and by perfecting the AR we can decrease $D[x]$ and $D[F]$, and improve the TEC of production.

Thus for example the amount of supplementary expensive elements that are added during steelmaking is determined in accordance with the requirements outlined in the GOST, there is a lowest, x_1, and highest, x_2, permitted

* If $F(x)$ has a minimum, then $M[F] > \bar{F}_{id}$. If the derivative $(dF(x)/dx)$ has a minimum when $F = F_{extrem}$, then either $M[F] = F'_{id}$ or $M[F] \gtrless F'_{id}$.

** In the general case $M[x] \neq x_g$.

percentage of these elements in the metal (Fig. 1). Usually we base the chemical control process on the value $M[x] = 0.5\,(x_1 + x_2)$. If we were to decrease the error involved in the measurement of the actual amount of a given element in steel and also the error involved in the weighing of the correcting supplements and study the amount that is destroyed in the oven then the stochastic variation in the amount of supplementary material in the final steel product is characterized by $D[x]$.

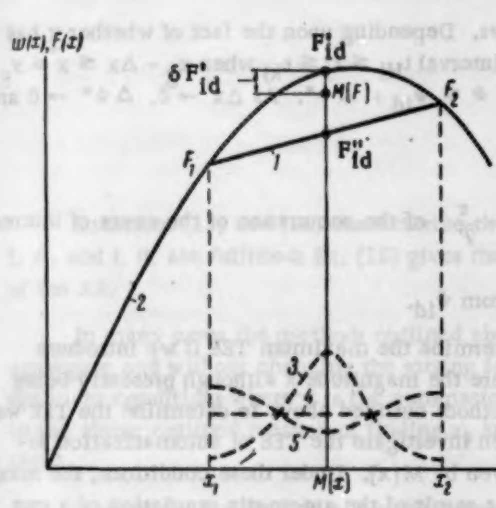

Fig. 1.

Under these conditions we can insure that $x_g < 0.5 \cdot (x_1 + x_2)$ and get as the cost price of the steel, $M[\Phi] < 0.5 \cdot [\Phi(x_1) + \Phi(x_2)]$. For $D[x] = \epsilon$, as $\epsilon \to 0$, $M[\Phi] = \Phi(x_1)$.

When no limitation is placed upon the values of $\underline{x}$ or $F(x)$, by changing the AR we can ensure that $M[x] = x_g$, then the conclusion reached above regarding the ineffectiveness of perfecting control devices for the control of the magnitude x remains valid.

3. Experimental Methods for Determining the Values of M[x], M[φ] and Φ_{id} for a Given AR

Let us examine the simplest type of technological process, the one where the variation of the TEC Φ is determined by stochastic changes in only one regulated or unregulated, inertial or inertialess variable $\underline{x}$.

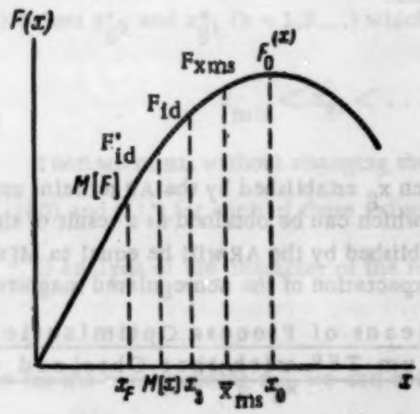

Fig. 2.

We can solve the problem of determining the values of Φ_{id}, $M[\Phi]$, $M[x]$, etc. for x_g = const, in various ways.

If we can determine the function $\Phi(x)$ (by calculation, correlation analysis, etc.) then, knowing x_g, we can find the value of Φ_{id}. Then if by using experimental methods we can determine the distribution $w(x)$, we can by means of Eqs. (3) and (6) calculate $M[x]$ and $M[\Phi]$.

We can also find the values of $M[x]$ and $M[\Phi]$ experimentally in the presence of a disturbance if we can measure

$$M[x] = \frac{1}{T}\int_0^T x\,dt, \tag{8}$$

$$M[\Phi] = \frac{1}{T}\int_0^T \Phi(x, t)\,dt, \tag{9}$$

where T is the period of time during which the process was observed.

Usually the function $\Phi(x)$ is unknown. In that case the value Φ_{id} can be determined experimentally by the creation of a device that will measure

$$\Phi_{id} = \frac{1}{\tau}\sum_{j=1}^{N}\int_{t_{nj}}^{t_{kj}} \Phi(x, t)\,dt, \qquad \tau = \sum_{j=1}^{N}(t_{kj} - t_{nj}) \tag{10}$$

for the condition

$$x_g - \Delta x \leqslant x \leqslant x_g + \Delta x. \tag{11}$$

Here $t_{kj} - t_{Hj}$ is the duration of the j-th interval, τ is the total length of the individual intervals in the process for which condition (11) holds, and Δx is the allowed deviation of $\underline{x}$ from x_g (from the error conditions involved in the determination of Φ_{id}).

The physical meaning of Eq. (10) and conditions (11) is as follows. Depending upon the fact of whether $\underline{x}$ has inertia or is inertialess during the change process, if during the time interval $t_{Hj} \leq t_j \leq t_{kj}$ when $x_g - \Delta x \leq \underline{x} \leq x_g + \Delta x$, the value of the TEC $\Phi(x)$ must lie within the limits $\Phi_{id} - \Delta\Phi' \leq \Phi \leq \Phi_{id} + \Delta\Phi''$. As $\Delta x \to 0$, $\Delta\Phi'' \to 0$ and $\Phi \to \Phi_{id}$.

The greater Δx the greater the probability $\int_{x_g - \Delta x}^{x_g + \Delta x} w(x)\,dx = \frac{\tau}{T}$ of the occurrence of the event of interest to us [characterized by Eq. (11)] and the greater the deviation of Φ from Φ_{id}.

The methods outlined above may be successfully applied to determine the maximum TEE if we introduce automatic regulation of the stochastically changing magnitude $\underline{x}$; where the magnitude $\underline{x}$ although presently being measured is not as yet subject to regulation. In order to apply the methods outlined above to determine the TEE we must first determined $M[x]$ experimentally, using Eq. (8). We will then investigate the TEE of automatization assuming that the AR will insure the attainment of the required data given by $M[x]$. Under these conditions, the maximum expected value of the TEC, $\Phi = \Phi_{id}$, which will be obtained as a result of the automatic regulation of $\underline{x}$ can be found from Eq. (10) if we place upon $\underline{x}$ the restriction

$$M[x] - \Delta x \leqslant x \leqslant M[x] + \Delta x. \tag{11'}$$

The difference

$$\delta\Phi_1 = \Phi_{id} - M[\Phi] \tag{12}$$

characterizes the maximum expected TEE for a perfect AR when the given x_g established by the AR remains unchanged. Equation (12) also permits us to determine the maximum TEE which can be obtained as a result of the application of an ideal AR for an unregulated value $\underline{x}$ when the data established by the AR will be equal to $M[x]$ where $M[x]$ is the experimentally-obtained value of the mathematical expectation of the nonregulated magnitude $\underline{x}$.

4. Determination of the Maximum TEE Obtained by Means of Process Optimization Using the Existing AR, and Comparison of this Maximum TEE with that Obtained with the Use of the Perfected AR

In the general case the desired value x_g, which is maintained by the AR, or the existing mathematical expectation $M[x]$ of the nonregulated magnitude may not correspond to the optimum TEC $M_0[\Phi]$. Under these conditions we would then consider it advisable to determine the following.

I. A. What maximum improvement in the TEC could be obtained by finding and using an optimum setting for the existing AR? (This optimum setting to be determined by any available means.)

B. What further maximum improvement in the TEC would be ensured by the use of an infinitely-perfected AR?

C. What would be the total TEE obtained as a result of the simultaneous fulfillment of conditions I. A. and I. B?

II. What maximum improvement in the TEC could be obtained by using an ideal AR with an optimum setting where this control would be applied to a magnitude which is not being regulated at the present time.

In order to solve the problem outlined in paragraph I. A. we must first determine the response of the existing AR to several given settings; by determining the response to each of many settings [x_{gk} = const ($k = 1, 2, \ldots$)] we can then, by means of Eq. (9), determine the value of $M[\Phi]$ associated with each setting. In this manner we can determine the character of the change in $M[\Phi]$ and find the optimum values x_0 and M_0 (see curve 2, Fig. 3).

If for $x_{gk} = x_0$, we can by using (10) and (11) determine Φ_{id}, then this value will correspond to $\Phi_0^{(x)}$ (see Fig. 3). In the general case, where the given or desired value is x_0, the corresponding $M_0[\Phi]$ and $\Phi_0^{(x)}$ may not coincide.

In this manner we can solve the problem outlined in paragraph I. B.

We can then calculate

$$\delta\Phi_A = M_0[\Phi] - M[\Phi], \tag{13}$$

$$\delta\Phi_B = \Phi_0^{(x)} - M_0[\Phi], \tag{14}$$

$$\delta\Phi_C = \Phi_0^{(x)} - M[\Phi]. \tag{15}$$

Equations (13) and (14) characterize the maximum TEE obtained when the conditions indicated in paragraphs I. A. and I. B. are fulfilled; Eq. (15) gives the maximum value of the TEE obtained by optimization and perfection of the AR.

In many cases the methods outlined above for determining Φ_{id} permit us, without interfering in the work of the aggregate and without changing the setting for the existing AR, to find the optimum mode of operation and the optimum conditions where $\underline{x}$ is the automatically regulated or nonregulated magnitude. The possibilities involved in the above outlined method of finding x_0 are based upon the assumption of stochastic deviations (x) from x_g or M[x].

The method of finding x_0 for the case where, for example, $\underline{x}$ is a nonregulated magnitude reduces to the following.

We determine experimentally the values of x_{min} and x_{max} for stochastic changes in $\underline{x}$. We then choose several values x'_{gk} and x''_{gk} (k = 1,2,...) which satisfy the condition

$$x_{min} < x'_{gk} < \ldots < x'_{g1} < M[x] < x''_{g1} < , \ldots , < x''_{gk} < x_{max}. \tag{16}$$

Then we must, without changing the mode of operation of the aggregate, experimentally determine, using Eqs. (10) and (11'), for each of these values of $\underline{x}$ the associated values of $\Phi_{id}^{(x'_{gk})}$, $\Phi_{id}^{(M[x])}$, $\Phi_{id}^{(x''_{gk})}$ (k = 1,2,...).

If analysis of the character of the function $\Phi_{id}(x_{gk})$ which we obtain shows that

$$x'_{gk} < x_0 < x''_{gk},$$

then for the corresponding x_{gk} we can obtain more precise values of x_0 and $\Phi_0^{(x)}$ by the method indicated above.

If we assume for the nonregulated magnitude $\underline{x}$ that $\delta\Phi_1 \approx \delta\Phi_B$, then using Eqs. (13)-(15) we can for this magnitude also determine the maximum TEE obtainable by using the AR and optimizing the process.

If for example $x_0 > x_{max}$, then we can in this manner determine for the nonregulated magnitude $\underline{x}$ only the value $\Phi_{id}^{(x''_{gk})} < \Phi_0^{(x)}$, where $x''_{gk} < x_{max}$ (see Fig. 3). If $\underline{x}$ is an automatically-controlled value, then if we establish a new mode for the AR given by x''_{gk} we can continue in our search and find x_0.

The application of digital differential analyzers permits us to simultaneously determine the value of Φ_{id} for the entire range $x_{min} < x < x_{max}$ and thus to basically accelerate the search process.

The duration of the process is limited only by the time required for the stochastically changing value of $\underline{x}$ to fall within the established limits $x_{gk} - \Delta x \leq x \leq x_{gk} + \Delta x$.

If for a constant setting x_g = const, the values of

$$\Phi_{id} = \frac{1}{t_{kj} - t_{nj}} \times \int_{t_{nj}}^{t_{kj}} \Phi(x, t)\, dt \quad (j = 1, 2, \ldots)$$

obtained for the various intervals are equal, then the existing variation in the TEC is determined only by the stochastically changing deviations of $\underline{x}$ from x_g. However a substantial difference between the values of Φ_{id} for adjacent

intervals of the process indicates that the TEC being studied depends upon the stochastic changes in a series of magnitudes. In our next paper we will examine the more complex process.

The method described above may also be used to analyze processes in which limitations are imposed upon the allowed values of $\underline{x}$ or F.

Assuming that due to technological considerations the limitation $x \leq x_{max}$ has been imposed upon the controlled magnitude $\underline{x}$, for existing AR this limitation is satisfied when $\int_{-\infty}^{\infty} x\, w(x)dx = M(x)$ (see curve 3, Fig. 3). Under these conditions only a perfected AR, designed to decrease the magnitude $x_{max} - M[x]$, opens up the possibility of improving the TEC of production. The expected improvement of the TEC of production can be calculated from Eq. (13).

In a series of cases when the conditions $\overline{x}_{ms} = x_{msg}$ or $F = F_g$, etc., are fulfilled we were able to improve our use of oven transformers (during heating), the inputs provided, and we were able to stabilize the amount of fuel or electrical energy supplied to the aggregate and therefore to improve the TEC of production.

The use of the simplest calculating devices or computers [1] has enabled us to correct the setting of the AR by means of the conditions

$$\int_0^t (x^2 - x^2_{msg})\, dt \to 0, \qquad \int_0^t (F - F_g)\, dt \to 0; \tag{17}$$

these enabled us, even for existing AR, to satisfy the conditions $\overline{x}_{ms} = x_{msg}$, $M[F] = F_g$, etc.

5. Calculation of the Maximum Permitted Capitalization to be used in the Improvement of AR or ACS

After the initial values that are to be used in Eqs. (12)-(15) have been determined experimentally we can calculate the amount of the product π produced during the year for the existing automatic control system (EACS) and for the perfected automatic control system (PACS), and also the maximum yearly economy in fuel, electrical energy, etc.

On the basis of the data we can for equal quantities of production reduced to π_{id} calculate for the EACS and the PACS the corresponding values of the annual production cost (C and C_{id}) and we can also determine the amount of additional (supplementary) capitalization ΔK required for the PACS as well as for the EACS for a change in production from π to π_{id}.

If the ideal AR of PACS for a certain production volume denoted by π_{id} will assure an economy of σW kilowatt hours in electrical energy and σG tons of fuel per year then we should take into account the capital expenses ΔK_n in the branches of industry associated with the construction of electrical power stations or with mines which are producing the above indicated amount of electrical energy, fuel, etc.

If we eliminate from the actual amortization time the time T_0 required for the associated branches of industry (given and considered)* then we can determine from our economic indices the maximum allowed size of the additional capitalization of the perfected AR of PACS

$$K_{id} = \Delta K + \Delta K_n + (C - C_{id})(1 - 0.01a)\, T_0, \tag{18}$$

where $\underline{a}$ is the amount of the amortization deductions.**

It is obvious that for actual conditions we cannot as a result of improvement of the AR or of optimization of the control processes reach the maximum indices which would be reached with an ideal AR or under ideal conditions existing with the optimum modes of operation.

6. Methods for the Approximate Evaluation of the TEC of the Aggregate when an Improved AR is used

By comparative study of the existing and proposed AR we can estimate the improvement in the control quality that would ensue as a result of the various degrees of improvement, of increased complexity, and of increased cost

* In conformance with [2], for most branches of industry T_0 = 3 to 7 years.

** The coefficient $\underline{a}$ was introduced following the advice of V. M. Dobkin.

of a proposed, improved (perfected) AR. We can conduct such an investigation by modelling, by combining the mathematical model of the object with the model of the proposed AR and with the existing AR, or by other means.

With the aid of known methods we can determine the probability distribution function $w(x)$ experimentally, using an existing AR on an industrial object.

For a steady, smooth technological process a relationship must exist between the probability distribution function $w(x)$ and the quality of the control process of the practicable AR regulating the magnitude $\underline{x}$. If this relationship can be established then, assuming that there are indices for the control quality of the existing and proposed AR, and also that we have experimentally obtained the function $w(x)$ for the existing AR, we can determine the function $w'(x)$ for the proposed AR and find the change in the TEC.

Let us consider the simplest example. Let us assume that by perfecting or improving the AR we can change the quality of the regulation process for the variable $\underline{x}$ so that when a disturbance or perturbation occurs, the curve giving the change in $\underline{x}$ due to the regulation process is compressed in half with respect to both the time and magnitude of the deviations (Fig. 4). We then get

$$\frac{x_{1m}}{x_{2m}} = \frac{x'_{1m}}{x'_{2m}} = \frac{t_1}{t_2} = \frac{t_{p1}}{t_{p2}} = 2.$$

Let us also assume that the experimentally obtained histogram of the distribution $f(x)$ for the regulated magnitude, regulated with the existing AR (shown as Curve 1, Fig. 5) has a symmetrical form. The dependence of the index $\Phi(x)$ shown in Curve 3, is symmetrical with respect to $M[x]$. We wish to determine the effectiveness of the proposed improvement in the AR.

If it is known that after reacting to each perturbation or disturbance the AR is passive and awaits the next disturbance then we can represent the histogram of the distribution $f(x)$ for the improved AR by Curve 2, Fig. 5, where $b = 0.5\,a$, $|x_2 - M[x]| = ab = 0.5\,\alpha a = 0.5\,|x_1 - M[x]|$, $f_{2(\alpha+1)} = 0.5\,f_{1(\alpha+1)}$, $f_{21} = f_{11} + 0.5 \sum_{\alpha=1}^{\infty} 2f_{1\,(\alpha+1)}$, where $\alpha = 1, 2,$

Assuming a histogram for the $f(x)$ distribution and a dependence $\Phi(x)$, it is not difficult to construct the corresponding histogram of the index distribution Φ for both regulators. In Fig. 5 these histograms are represented by Curve 1' for the existing AR and Curve 2' for the desired or proposed improved AR. We can now calculate the mathematical expectations $M[\Phi_1']$ and $M[\Phi_2']$ shown in Fig. 5. The difference $M[\Phi_2'] - M[\Phi_1']$ characterizes the improvement in the TEC under study associated with improvement of the AR and $\Phi_0^{(x)} - M[\Phi_2']$, the improvement in the TEC associated with the transition from the improved AR to the ideal AR.

On the basis of Fig. 5 we can come to a series of important conclusions regarding some peculiarities of the $\Phi(x)$ distribution.

1. In the general case the distribution curve of the TEC $\Phi(x)$ is unsymmetrical.

2. The maximum value of Φ in the distribution curve of the index $\Phi(x)$ is Φ_{max}. It cannot exceed $\Phi_0^{(x)}$.

3. In accordance with paragraph 2 we find that the mathematical expectation $M[\Phi] < \Phi_0^{(x)}$.

On the basis of these conclusions we can propose another method for experimentally determining the expected effectiveness of an improvement in the EACS based upon a study of the distribution curves $w[\Phi]$ which takes their characteristic properties into account. The advantages of this method of investigation are that the length or duration of the study are decreased and also we obtain more complete information regarding the TEC of the process under study.

The essence of this method consists of studying the $\Phi(x)$ distribution. If the $w(x)$ distribution includes the value of x_0 then the maximum value of $\Phi_0^{(x)}$ which may be obtained experimentally is $\Phi_0^{(x)}$ and the probability of obtaining this value is greater the larger $w(x_0)$.

This situation is illustrated in Fig. 6. For a choice of x_g corresponding to curves 1 and 1', $\Phi_{1'max} < \Phi_{max}^{(x)}$.

Changes in x_g which insure a probability of observing a value of $x = x_0$, equal to $w(x_0)\,\Delta x > 0$ (characteristic distribution curves 2, 2' and 3, 3'), permit us to obtain the value $\Phi_{max} = \Phi_0^{(x)}$. An increase in x_g (operation corresponding to curve 3) leads, when compared to an operation corresponding to curve 2, merely to an increase in the probability of observing the value $\Phi_{max} = \Phi_0^{(x)}$.

7. Determination of the Economically Expedient Limits of Improvement of a Control System

As a result of the improvement of AR or other control devices we might be able to improve the TEC of production by an amount $\delta\Pi$, δW, δC, etc., as outlined above $\delta\Pi < \delta\Pi_{id}$, $\delta W = \delta W_{id}$, etc. The economically permissible capitalization that is available for the improvement of each AR or other control device is $K_{pr} < K_{id}$.

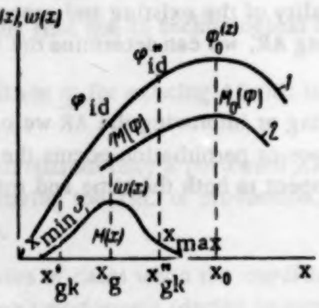

Fig. 3.

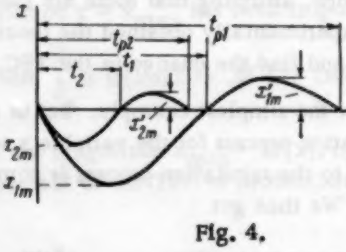

Fig. 4.

Let us assume that we have with the aid of the methods outlined above determined the value of K_{pr} for EACS with varying degrees of improvement and that we have also found the value of K_{id}. Let the cost of building and establishing a certain type of EACS to K_c and the cost of an improved system of this type PACS be K_y. Then if we build the PACS rather than the EACS the additional required capitalization will be

$$K = K_y - K_c.$$

We will assume that the additional permissible capitalization is K_{pr}.

In Fig. 7 we plot the values of K_{pr} and K_{id} along the axis of abscissas and the values of K along the axis of ordinates.

We will also plot the amortization time for the additional capitalization T in Fig. 7. We determine the scale used in plotting the variable T from the condition for the required additional capitalization $K = K_{id}$, $T = T_0$. We will plot line 1. The equation for line 1 is $K = K_{pr}$. Point d on the line $K = K_{id}$ corresponds to the value $T = T_0$. It is obvious that for $K = 0$ and $K_{pr} \leq K_{id}$, $T = 0$. On the basis of these conditions we determine the scale for T_0.

If in Fig. 7 we draw a series of lines from the origin of coordinates to the points a, b,... e, corresponding to the values $T = 1, 2, \ldots, T_0 + 1$, then they will divide all the improved AR into groups based upon the values of K, K_{pr} and the required amortization times T, $T = K/K_{pr}\, T_0$, of the additional expenditures for the varying degrees of improvement required for each of the improved AR, the T's being less than $1, 2, \ldots, T_0 - 1, \ldots, T_0 + 1$ years, respectively. The condition for an economically permissible improved AR system is $T \leq T_0$. We can outline the boundaries of the region for the allowed values of K and K_{pr} by dashed lines.

Let us assume that in developing an improved AR we consider three different qualities of the control process, three different variants of the improved AR, these variants differing in degree of complexity and cost. The additional capitalization K actually required for the three variants under consideration and the limiting allowed capitalization K_{pr} are indicated in Fig. 7 by the points P_2, P_{T_0-1} and P_{T_0}. Let us pass curve 2 through these three points and through the origin of coordinates. The function $K(K_{pr})$ thus obtained represents the following law. Each successive step in the improvement of the AR directed towards obtaining values of K_{pr} approaching K_{id}, is obtained by an expensive (significant) progressive increase in actual supplementary capitalization K and in an increase in the amortization time required for these increases in capitalization. Thus, for example, for the three variations in AR characterized by points P_2, P_{T_0-1} and P_{T_0} for $T_0 = 2.5$ years the corresponding amortization times for these expenditures are 1, 2, and 2.5 years respectively. If the greatest degree of improvement in the control quality and in the value of K_{pr} (in comparison with the variation of the AR characterized by the point P_{T_0}) can be realized only for $T > T_0$ we can in the general case consider that a greater degree of perfection (improvement) of the AR is not economically expedient.

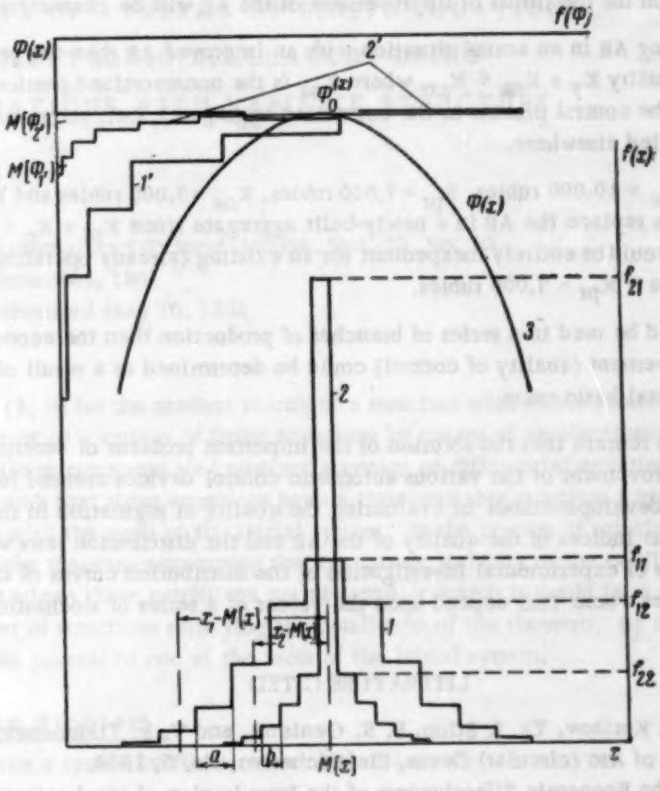

Fig. 5.

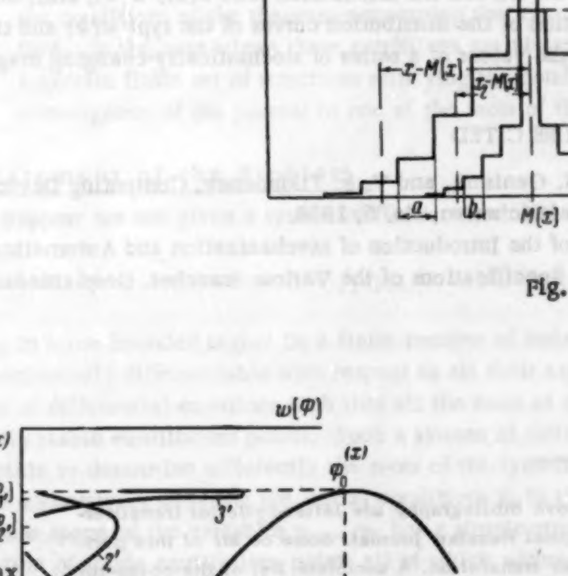

Fig. 6.

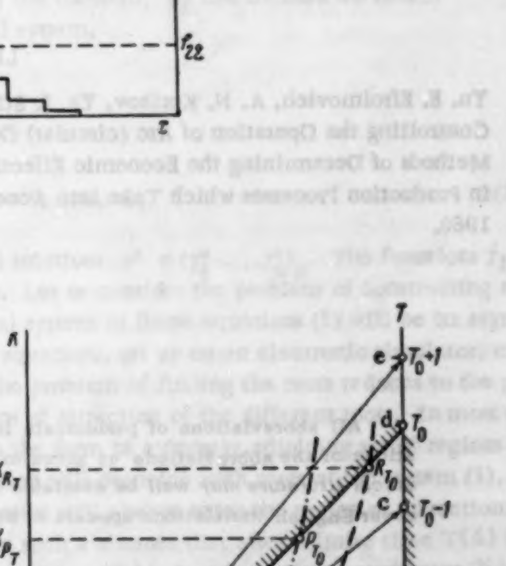

Fig. 7.

With the emergence of new technical automatization processes and changes in the other processes or conditions described above it might turn out that relations K and K_{pr} will be characterized by curve 3 (Fig. 7). In that case the economic justification for the limits of improvement of the AR will be characterized by the point R_{T_0}.

If we replace the existing AR in an actual situation with an improved AR then the permitted value K_y must be determined from the inequality $K_y + K_{na} < K_{pr}$, where K_{na} is the nonamortized portion of the cost of the AR itself (the AR removed from the control process or the controlled aggregate), i.e., removed from the control loop, which cannot be usefully applied elsewhere.

Thus, for example, if K_c = 10,000 rubles, K_{pr} = 7,000 rubles, K_{na} = 3,000 rubles and K_y 12,000 rubles, then it would be entirely expedient to replace the AR in a newly-built aggregate since $K_{pr} + K_c$ = 17,000 rubles > K_y = 12,000 rubles. However, this would be entirely inexpedient for an existing (already operating) aggregate in general since $K_y + K_{na}$ = 15,000 rubles > K_{pr} = 7,000 rubles.

If the improved AR could be used in a series of branches of production then the economic justification for the degree of perfection or improvement (quality of control) could be determined as a result of an analysis of the improvement of the TEC for several basic users.

In conclusion we wish to remark that the solution of the important problem of determining the economic expediency of the degree of improvement of the various automatic control devices created for the various technological processes requires that we develop methods of evaluating the quality of regulation in the system AR-object, determine the laws connecting the indices of the quality of the AR and the distribution laws w(x), w(Φ), etc., and also develop apparatus and methods of experimental investigation of the distribution curves of the type w(Φ) and the hypersurfaces $\Phi(x_i)$ for processes whose TEC depend upon the values of a series of stochastically-changing magnitudes.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

FINDING THE ROOTS OF SYSTEMS OF FINITE EQUATIONS BY MEANS OF AN ELECTRONIC SIMULATOR, USING DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE

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An idea of Ashby [1, 3] for the random search of a structure with stable motions is applied to the problem of finding a root of a system of finite equations by means of an electronic simulator. From the initial system of finite equations we construct a system of differential equations suitable for setting up on the simulator such that these equations have a time-variable structure with stationary equilibrium points coinciding with the roots of the initial system. In the process of solution a check is made that the conditions of the theorem concerning the asymptotic stability of the equilibrium point are satisfied. In the case where these conditions are violated, a search is made for a suitable structure out of a certain finite set of structures satisfying the conditions of the theorem. By this method we obtain convergence of the process to one of the roots of the initial system.

1. Statement of the Problem

Suppose we are given a system of finite equations

$$f_i(y_1, \ldots, y_n) = 0 \qquad (i = 1, \ldots, n), \tag{1}$$

having in some bounded region D_1 a finite number of isolated real solutions $y^* = (y_1^*, \ldots, y_n^*)_k$. The functions f_i are continuously differentiable with respect to all their arguments. Let us consider the problem of constructing a system of differential equations such that all the roots of the initial system of finite equations (1) will be its asymptotically stable equilibrium points. Such a system of differential equations, set up on an electronic simulator, makes it possible to determine efficiently the roots of the system (1). The problem of finding the roots reduces to the problem of successively situating the initial conditions y_0 in the regions of attraction of the different roots. In most cases the phase space of the variables $y_1, \ldots, y_n$ has a simple structure in the form of a densely adjoining set of regions of attraction of stable equilibrium points all of which without exception will coincide with roots of the system (1). In these cases for given initial conditions $y_0 \in D$, the representative point will always enter the region of attraction of one of the equilibrium points y^* and will thereafter approach it in such a manner that after a finite time $T(\delta)$ it will enter any predetermined δ-neighborhood of it. For sufficiently large $T(\delta)$, the values of the coordinates $T(\delta)$ $y_1(T), \ldots, y_n(T)$ in the scheme to be set up may be taken as the values of the coordinates of one of the roots of the initial system (1). By giving different initial conditions from the region D' we can successively enter the regions of attraction of different equilibrium points and thus find all the roots.

We shall construct the desired system of differential equations in the form

$$\frac{dx_i}{dt} = F_i(f_1, \ldots, f_n, \varepsilon_1, \ldots, \varepsilon_r) \qquad (i = 1, \ldots, n). \tag{2}$$

Here $\epsilon_1(t), \ldots, \epsilon_r(t)$ are step-functions of time (Fig. 1), assuming values belonging to a certain set E, with $\epsilon_i(\bar{t}_k) = \epsilon_i(\bar{t}_k - 0)$ at the points of discontinuity; the functions F_i are continuous with respect to all f_i and ϵ_k, and, in addition, for any i the equations $F_i(0, \ldots, 0, \epsilon_1, \ldots, \epsilon_r) = 0$ and $F_i(f_1, \ldots, f_n, \epsilon_1, \ldots, \epsilon_r) \neq 0$ are valid if at least one of the functions f_i is different from zero. By $x_1, \ldots, x_n$ in the system (2) and throughout the following pages we mean the distances from some fixed equilibrium point, which, as is obvious, corresponds to one of the roots of

the system (1). The functions $\epsilon_1(t), \ldots, \epsilon_r(t)$ are introduced into the right-hand parts of the system (2) to make it possible for them to vary stepwise depending on whether the conditions of some stability criterion for the equilibrium points is satisfied.

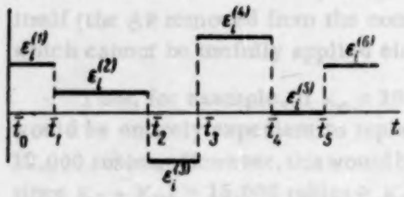

Fig. 1.

We form a positive definite function V of $f_1, \ldots, f_n$, that is, a function which is everywhere positive except for the points $f_1 = f_2 = \ldots = f_n = 0$, where $V(f_1, \ldots, f_n) = 0$. In a small neighborhood of the point $y_1^*, \ldots, y_n^*$, where $V = 0$, the family $V = C$ is a one-parameter family of closed surfaces enclosed within one another. Hereafter we shall consider all the solutions of the system (2) in the region $D \subset D'$ including such a point and having the structure of a small neighborhood. The total derivative of the function V with respect to $\underline{t}$, by virtue of the system (2), is equal to

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i\,(f_1, \ldots, f_n, \varepsilon_1, \ldots, \varepsilon_r).$$

(The derivatives with respect to $\underline{t}$ are taken from the left, and by virtue of the condition $\epsilon_i(\bar{t}) = \epsilon_i(\bar{t}_k - 0)$, $\dot{\epsilon}_i \equiv 0$, $i = 1, \ldots, n$.)

The point y^* will be asymptotically stable for initial conditions $y_0 \in D$, if in the entire infinite interval $[t_0, +\infty]$ the inequality

$$\dot{V} \leqslant -W \tag{3}$$

is satisfied, where W is a positive definite function depending on the local phase coordinates $x_1, \ldots, x_n$. If the inequality (3) is satisfied, the conditions of Lyapunov's theorem [8] on asymptotic stability will be satisfied. In particular, the point $x^* = 0$ will be asymptotically stable if the inequality (3) is satisfied for

$$W = \alpha V, \quad \alpha > 0. \tag{4}$$

We assume that the functions F_i are so chosen that for each point $(x_1, \ldots, x_n) \in D$ there exists such a set $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ included in E that (3) is satisfied. Under these conditions the following method is suggested for constructing the functions $\epsilon_1(t), \ldots, \epsilon_r(t)$ which ensure that the condition (3), and, consequently, the conditions of asymptotic stability, will be satisfied. At the initial point x_0 we select in an orderly or random fashion a set $\epsilon_1^{(1)}, \ldots, \epsilon_r^{(1)}$, for which the inequality (3) will be satisfied. Furthermore, let a solution of the system (2) continue with these fixed parameters up to a time t_1 such that from this time on the relations

$$\begin{aligned} \dot{V} &= -W\,(x_1\,(t_1), \ldots, x_n\,(t_1)), \\ \dot{V} &> -W\,(x_1\,(t), \ldots, x_n\,(t)) \qquad (t > t_1) \end{aligned} \tag{5}$$

are satisfied.

At the point t_1 we select a new set of numbers $\epsilon_1^{(2)}, \ldots, \epsilon_r^{(2)}$ (by hypothesis such a set exists), such that for $t > t_1$ the inequality (3) will again be satisfied, and so on. Lyapunov's theorem, which requires that the strong inequality (3) be strictly satisfied, at the same time provides only sufficient conditions for asymptotic stability. Therefore, in the case where it is not possible to satisfy the inequality (3) on the whole trajectory, we may require that a weaker condition be satisfied, namely, the inequality

$$\frac{dV}{dt} \leqslant 0, \tag{6}$$

where the equality sign is allowable only for isolated finite intervals of time.

Under certain additional conditions of a general nature which satisfy the requirement $V \to 0$ as $t \to +\infty$ [8], the equilibrium point will be asymptotically stable. In practice, in the absence of limit cycles and certain other special phenomena, the required decrease of the function V is guaranteed if the inequality (6) is satisfied. Hereafter, when we construct concrete systems, besides the inequality (3), we shall also use the inequality (6). We shall make a search for the numbers $\epsilon_1, \ldots, \epsilon_r$ each time that

$$\frac{dV}{dt} > 0. \tag{7}$$

When the above-mentioned method of selecting the functions ϵ_k is used, two types of trajectories may be encountered. In the first place, (Fig. 2a) after switching the functions ϵ_k a finite number of times the trajectory asymptotically approaches an equilibrium point without any further switching, that is, there exist fixed sets of values $\epsilon_1,\ldots,\epsilon_r$ such that the equilibriumpoint of the system (2) is asymptotically stable, at least in the small (in Fig. 2a, $\overline{D}$ is the region of asymptotic stability). In the second case (Fig. 2b) the number of switchings is infinite, that is, the equilibrium point of the system (2) is unstable (even in the small) for every set of values $\epsilon_1,\ldots,\epsilon_r$ belonging to E, or else stable in the Lyapunov sense.

In practice, since there is a zone of insensitivity in the indicator which determines whether the inequality (3) is satisfied, the motion in the neighborhood of the equilibrium point may differ from the theoretical motion; for example, it may be oscillatory. For small values of $\dot{V}$ the amplitude of the oscillations of the function V, and consequently the amplitude of the oscillations of the phase coordinates as well, may be considerable, which reduces the accuracy of calculation of the coordinates of the equilibrium point. For this reason it is preferable, if possible, to construct systems with the first type of motion.

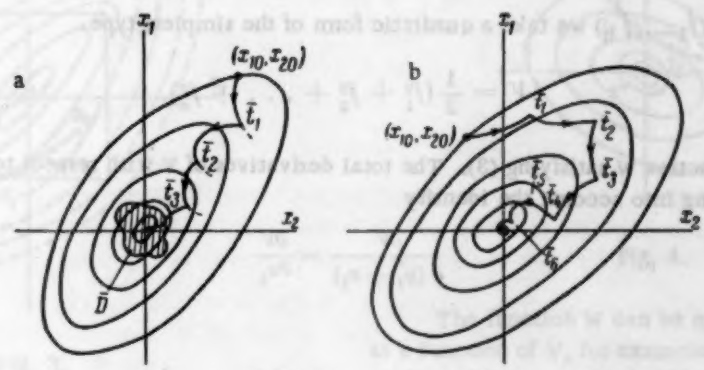

Fig. 2.

In the system instrumented on the simulator, other types of motion are possible as well. Thus, if the region L, bounded by the natural linearity limits of the computing elements and having the form of a hyperparallelepiped, can be completely inscribed in D, then it is possible for the representative point to go out to the boundary of the region L (Fig. 3), where a system of differential equations other than the system (2) will become operative and the point x_1 may be found to be a stable equilibrium point. In this case a choice of new initial conditions is necessary. It is possible also to have a second variant, when from the point x_1 the motion proceeds along the boundary of the region L with a subsequent departure from the point x_2, where $\dot{V} < 0$, to the interior of the region. Finally, in a real system it is possible to have divergent motion (Fig. 4). If the condition (3) is satisfied, the representative point x(t) will intersect the level surfaces from the outside toward the inside.

As a result of the inaccuracy of determining the moment of switching of the parameter values $\epsilon_1,\ldots,\epsilon_r$ and the finiteness of the time required to search for them, the motion may deviate greatly from the theoretical motion. Suppose the switching takes place not at the point t_2, where $\dot{V} = 0$, but at the point t_3. At this point $V_3 > V_2$. This means that as a result of this motion the representative point has come out on a surface with a higher value than that from which the motion began, that is, the representative point for such segments of the trajectory will intersect the level surfaces from the inside toward the outside. The motion will be divergent.

In all of the above considerations it was supposed that the initial conditions are taken strictly in the region D, where the trajectories converge to an equilibrium point, which is a solution of the initial system (1). However, for arbitrary initial conditions it may happen that the equilibrium point will not be a root of the system (1). Such a case may happen if the function V has relative minima, that is, minima at which $V \neq 0$, and, consequently, the equation $f_i = 0$ is not satisfied for all f_i. It should be noted that in the neighborhood of such a point we will find an infinite and fruitless process of search for the numbers $\epsilon_1,\ldots,\epsilon_r$ for which the conditions (3) or (6) will be satisfied. In this case we must halt the simulator and renew the search with initial conditions sufficiently far from this point.

Here, as in the use of the gradient method [10, 11], we are faced with the problem of constructing a function V of $f_1,\ldots,f_n$ with no relative minima. It can be proved (see Appendix) that if the Jacobian of the system (1) does not

vanish anywhere in the region D, then the function

$$V = \lambda_1 f_1^2 + \lambda_2 f_2^2 + \ldots + \lambda_n f_n^2, \qquad \lambda_i > 0 \qquad (i = 1, \ldots, n) \tag{8}$$

has no relative minima in this region. In the general case this problem has not been solved.

2. Solution of the Problem for one Class of Finite Equations

Let us consider the problem of finding a root of a system of finite equations for which the Jacobian is nonzero at every point of the region D. For example, such a system may be constituted by a system of linear algebraic equations with a nonzero determinant.

For this purpose we shall construct a system of differential equations (2) of the form

$$\frac{dx_i}{dt} = \varepsilon_i, \qquad V(f_1, \ldots, f_n), \tag{9}$$

where for the function $V(f_1, \ldots, f_n)$ we take a quadratic form of the simplest type.

$$V = \frac{1}{2}(f_1^2 + f_2^2 + \ldots + f_n^2). \tag{10}$$

We shall find a function W, satisfying (3). The total derivatives of V with respect to $\underline{t}$, by virtue of the system of equations (9) and taking into account the identity

$$\frac{\partial V}{\partial (y_i^* + x_i)} = \frac{\partial V}{\partial x_i}$$

will have the form

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = V \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} f_k \right) \varepsilon_i. \tag{11}$$

It can be proved that if the Jacobian $\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}$ does not vanish in the region D, except for the solution points x^*, then at least one of the partial derivatives $\partial V/\partial x_i$ is nonzero in this region. To convince ourselves of this, let us observe any point different from x^*. At this point at least one of the functions $f_i \neq 0$. Referring to (11), we can write

$$\frac{\partial V}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} f_i = 0.$$

Considering the last system of equations as a system with a nonzero solution with respect to the variables $f_1, \ldots, f_n$ and a Jacobian different from zero, we arrive at a contradiction. Consequently the equations $\partial V/\partial x_i = 0$ are not possible for all $\underline{i}$ simultaneously at any point except the point x^*, where $f_1 = f_2 = \ldots = f_n = 0$.

Let us take as the function W the following function:

$$W = \beta V \sum_{i=1}^{n} \left| \frac{\partial V}{\partial x_i} \right| \qquad (0 < \beta \leqslant 1). \tag{12}$$

This function does not depend on $\underline{t}$, is everywhere positive, and becomes zero at the same point as does the function $\dot{V}$, that is, at the point y^* or $\underline{x}^* = 0$ (in local coordinates). Furthermore, the condition (3) will be satisfied if we set*

$$\varepsilon_i = -1 \ \text{sign} \ \frac{\partial V}{\partial x_i} \qquad (i = 1, \ldots, n).$$

---

* sign a = 1 for a > 0 and sign a = −1 for a < 0.

These equations can always be satisfied by a suitable choice of the numbers $(\epsilon_1,\ldots,\epsilon_n) \in \{+1,-1\}$, and consequently it is always possible to have the inequality

$$\dot{V} \leqslant -W.$$

Thus, the problem will be solved if as the function W we take the function (12). However, for the formation of the latter, in a number of problems we may require in addition a considerable number of function generators, particularly if the functions f_i are transcendental.* It is therefore desirable that the function W should depend on the functions necessary for the formation of the system (9). For example, the functions $f_1,\ldots,f_n, f_1^2,\ldots,f_n^2$, V are such functions.

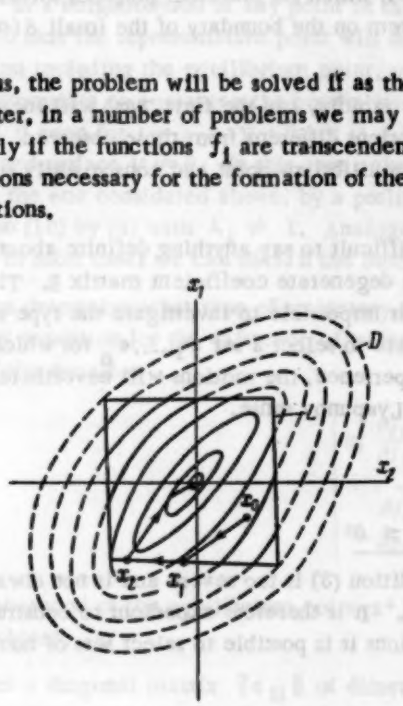

Fig. 3.

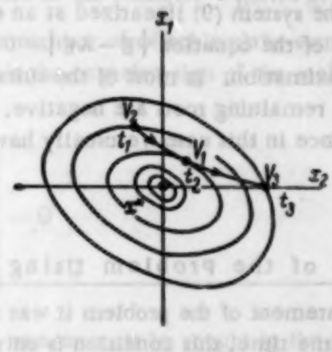

Fig. 4.

The function W can be most simply instrumented as a function of V, for example, in the form (4). Let us analyze this case. We should remark that for a suitable set of numbers $\epsilon_1,\ldots,\epsilon_n$ the maximum value of the phase velocity V for fixed $\underline{t}$ is given by (12) for $\beta = 1$ and the condition (3), in view of (4), will take the form

$$V\sum_{i=1}^{n}\left|\frac{\partial V}{\partial x_i}\right| \leqslant -\alpha V.$$

Taking into consideration the fact that for $x \neq 0$, $V \neq 0$, we obtain

$$\sum_{i=1}^{n}\left|\frac{\partial V}{\partial x_i}\right| \geqslant \alpha, \quad \alpha > 0.$$

From the last relation it follows that as the representative point approaches $x^* = 0$, by virtue of continuous dependence of the left side of this relation on $\underline{x}$, there will arrive an instant of time $\tilde{t}$, for which

$$\sum_{i=1}^{n}\left|\frac{\partial V}{\partial x_i}\right| = \alpha, \tag{13}$$

and for $t > \tilde{t}$ the inequality sign in this relation changes direction and will not change thereafter for any set $\epsilon_1,\ldots,\epsilon_n$.

*The partial derivatives, as a rule, cannot be formed by the summation of signals from different points of the circuit, in view of their fundamental difference from the component f_i of the function. For example, if $f = \ln x + \tan x + b$, then $\frac{\partial V}{\partial x} = \frac{1}{x} + \frac{1}{\cos^2 x}$, and thus we cannot use the component terms of the function f to form the function $\partial V/\partial x$. In the case where f is an algebraic function, at least some of its terms will coincide to within the coefficients with the component terms of the function W. These terms can be used in setting up the function W.

Thus, in principle, the representative point will not reach the equilibrium position $\underline{x}^* = 0$ for any finite $\alpha > 0$. However, for sufficiently small α, in accordance with Eq. (13), the representative point will enter an arbitrarily small neighborhood $\delta(\alpha)$ of the point $\underline{x}^*$ after a finite length of time $T(\alpha)$, which for practical purposes solves the problem under consideration. In practice, oscillations will appear in the system on the boundary of the small $\delta(\alpha)$ neighborhood.

For $\alpha \to 0$ we may encounter cases in which V will decrease rapidly, and the time $T(\alpha)$ will increase rapidly, so that a simple transition from the relation (3) to (7) may cause motions different from those observed. Thus, choice of function W in the form (4) practically assures convergence to an equilibrium point and consequently solves the problem stated.

In conclusion we remark that in using the function (12) it is difficult to say anything definite about the type of trajectory. The system (9) linearized at an equilibrium point has a degenerate coefficient matrix B. Therefore one of the roots λ_i of the equation $|B - \lambda E| = 0$ is a zero root, and it is impossible to investigate the type of motion by a first approximation. In most of the suitable cases, if it is possible to select a set $\epsilon_1, \dots, \epsilon_n$ for which the real parts of all the remaining roots are negative, then as is shown by experience, the motions will nevertheless be of the second type, since in this case we usually have only stability in the Lyapunov sense.

3. Solution of the Problem Using the Criterion $\dot{V} \leq 0$

In the statement of the problem it was remarked that the condition (3) is too severe and is not always satisfied. At the same time, this condition is only a sufficient condition. It is therefore expedient to construct systems of differential equations in which for most of the conceivable situations it is possible to select sets of numbers $\epsilon_1, \dots, \epsilon_r$ for which

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i (f_1, \dots, f_n, \varepsilon_1, \dots, \varepsilon_r) \leqslant 0.$$

In most cases in such systems we have convergence of the process. Let us consider two such systems of differential equations. We shall switch values each time the inequality $\dot{V} > 0$ occurs. As the function V we shall take the function (10). The first of these systems has the form

$$\frac{dx_i}{dt} = \varepsilon_i f_i \quad (i = 1, \dots, n). \tag{14}$$

The functions $\epsilon_1, \dots, \epsilon_n$ are selected from a finite set E; in particular, the set $\{+1, -1\}$ may be such a set, and we shall restrict our attention to this. Each time that $\dot{V} > 0$, we shall change the signs of the functions f_i by a suitable choice of $\epsilon_i = 1$ or $\epsilon_i = -1$ until $\dot{V}$ changes sign. We shall show that this can be done in most cases. For this purpose we refer to the expression for $\dot{V}$:

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \varepsilon_i f_i. \tag{15}$$

Let us consider the function $\dot{V}$ at the moment $\bar{t}$, when $\dot{V} = 0$ and $\dot{V} > 0$ for $t > \bar{t}$. Two cases may occur:

1. At least two terms P and Q of the sum (15) are different from zero. Let

$$P = \frac{\partial V}{\partial x_k} \varepsilon_k f_k > 0 \text{ and } Q = \frac{\partial V}{\partial x_l} \varepsilon_l f_l = -P < 0.$$

Then, changing the sign of P, we obtain $\dot{V} < 0$ for $t > \bar{t}$.

2. All the terms of the sum (10) are equal to zero:

$$\frac{\partial V}{\partial x_i} f_i = 0. \tag{16}$$

If the equality (16) is satisfied at one point $(\bar{x}_1,\ldots,\bar{x}_n)$ or even on a manifold with dimension $m < n$, for example, on the surface $V = C$, then in practice this case reduces to the first. In a real simulator, when there are fluctuations, the representative point cannot be strictly on a manifold with a dimension less than the dimension of the space. In a neighborhood of any point of this manifold there exist regions where $\dot{V} \neq 0$. Therefore it should be expected that the representative point will not remain on such a manifold. It is only if $\dot{V} \equiv 0$ in an n-dimensional region including the equilibrium point, or in a layer bounded by the surfaces $V = C_1$ and $V = C_2$, $C_1 \neq C_2$, that it is impossible to guarantee the possibility of obtaining a negative sign for $\dot{V}$. Finally, we may have cases where $\dot{V} < 0$ even for $t \to +\infty$ $\dot{V} \to 0$ but $V \to C \neq 0$, that is, the representative point tends to a limit cycle located on some fixed surface $V \neq 0$. In this case only a stable limit cycle is of interest. However, even this case may be reduced to the one considered above, by a preliminary deformation of the surface $V = C$, for example, by replacing the function (10) by (8) with $\lambda_i \neq 1$. Analogous considerations are also possible for the system (9). Thus we conclude that in most cases we can obtain the inequality $\dot{V} < 0$ by a choice of the numbers $\epsilon_1,\ldots,\epsilon_n$.

Let us determine what type of trajectory (with a finite or infinite number of switchings) the system (14) has. A necessary condition for the existence of trajectories of the first type is the negative sign of the real parts of the roots λ_i of the equation

$$\begin{vmatrix} \varepsilon_1 \dfrac{\partial f_1}{\partial x_1} - \lambda & \ldots & \varepsilon_1 \dfrac{\partial f_1}{\partial x_n} \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ \varepsilon_n \dfrac{\partial f_n}{\partial x_1} & \ldots & \varepsilon_n \dfrac{\partial f_n}{\partial x_n} - \lambda \end{vmatrix} = 0$$

for a neighborhood of the equilibrium point $x^* = 0$. Consequently the answer must be sought in the solution of another problem.

Given a diagonal matrix $\|\epsilon_{ii}\|$ of dimension $\underline{n}$ and a square nonsingular matrix $\|a_{ij}\|$ of the same dimension. The question arises as to whether it is possible $\underline{\text{to}}$ choose a set of numbers $\epsilon_1,\ldots,\epsilon_n = \pm 1$ such that all the characteristic numbers λ_i of the matrix

$$\|c_{ij}\| = \|\varepsilon_{ii}\|\,\|a_{ij}\|$$

will have negative real parts. For a matrix $\|c_{ij}\|$ of triangular form, with $a_{ii} \neq 0$, this is always possible. The same can be said with respect to a complete matrix of dimension 2.

For $n > 2$, depending on the coefficients of the matrix $\|a_{ij}\|$, trajectories of either the first or the second type may exist.*

The very simple system considered above may be considerably improved. Instead of the system (14) let us consider the system

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \vdots \\ \dfrac{dx_n}{dt} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \ldots & \varepsilon_n \\ \ldots & \ldots & \ldots \\ \varepsilon_{n1} & \ldots & \varepsilon_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{17}$$

Here the diagonal matrix $\|\epsilon_{ii}\|$ of the system (5) is replaced by a complete matrix $\|\epsilon_{ij}\|$, where the ϵ_{ij}, as before, are real numbers belonging to a finite set. In this system for each search operation up to n^2 numbers may be chosen. The number of switching circuits will become $\underline{n}$ times as large, but the manner of computing amplifiers will remain practically the same as before. The new system (17) differs from the system (14) in that all of its convergent phase trajectories and trajectories of the first type.

* We can indicate a criterion which, when it is satisfied, always guarantees the existence of such a set $\epsilon_1,\ldots,\epsilon_n$. In [9] it is shown that if for a matrix $\|a_{ij}\|$ the two inequalities $a_{ii} < 0$, $|a_{ii}| > \Sigma |a_{ij}|$, $i \neq j$, are satisfied, then for this matrix $\text{Re}\,\lambda_i < 0$. Obviously, if only the second inequality is satisfied, we can always obtain negative diagonal elements for the matrix $\|c_{ij}\|$ by a suitable choice of $\|\epsilon_{ii}\|$.

In order to convince ourselves of this, it is sufficient to consider the linear approximation of the system (17) in a neighborhood of the equilibrium point $x^* = 0$:

$$\left\|\frac{dx}{dt}\right\| = \|\varepsilon_{ij}\| \left\|\frac{\partial f_i}{\partial x_j}\right\| \|x\| = \|c_{ij}\| \|x\|.$$

Here $\left\|\frac{\partial f_i}{\partial x_i}\right\|$ is the Jacobian matrix with elements $\frac{\partial f_i}{\partial x_j}$, calculated at the equilibrium point, and $\|\epsilon_{ij}\|$ are column matrices. Obviously there always exists a matrix $\|\epsilon_{ij}\|$ for any given matrix $\|c_{ij}\|$ such that

$$\|\varepsilon_{ij}\| = \|c_{ij}\| \left\|\frac{\partial f_i}{\partial x_j}\right\|^{-1}.$$

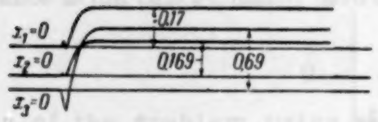

Fig. 5.

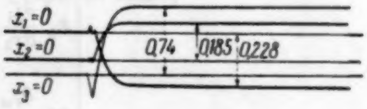

Fig. 6.

The matrix $\left\|\frac{\partial f_i}{\partial x_j}\right\|^{-1}$ is the inverse of the Jacobian matrix. The inverse matrix exists, since the Jacobian matrix is nonsingular in a neighborhood of the solution $x^* = 0$. Specifying the matrix $\|c_{ij}\|$ with roots, for which Re $\lambda_i < 0$, we obtain the matrix $\|\epsilon_{ij}\|$.

A number of problems were solved on the ÉMU-8 and MN-7 simulators. The results of the solution of one of these are given below. We solved a system of nonlinear algebraic equations:

$$\begin{aligned} f_1 &= x_1^2 - 0.01\,[4.16 - 3.73x_2 - (x_1 + x_2 + x_3)^2] = 0, \\ f_2 &= x_2^2 - 0.01\,[4.16 - 3.73x_2 - (x_1 + x_2 + x_3)^2] = 0, \\ f_3 &= x_3^2 - 0.17\,[4.16 - 0.93x_2 - (x_1 + x_2 + x_3)^2] = 0. \end{aligned} \tag{18}$$

A system of differential equations was set up on the simulator in the form (14)

$$\dot{x}_i = \varepsilon_i f_i \quad (i = 1, 2, 3, \quad \varepsilon_i = \pm 1). \tag{19}$$

As the function* V we took the function $V = c(f_1^2 + f_2^2 + f_3^2)$, $c > 0$. The total derivative $\dot{V}$ was formed by means of a differentiator. From the differentiator the signal was transmitted to the sign unit, at the output of which a polarized relay with a bias was connected. For $\dot{V} > 0$ the relay contacts connected the input of the trigger circuit to a generator producing sinusoidal oscillations with frequencies of 5-15 cycles. Each trigger was controlled by a polarized relay which changed the sign of the right-hand part of one of the equations of the system (19).

The oscillograms in Figs. 5 and 6 illustrate the search for two different roots. The relative error in finding the roots in all the tests is about 2%. The error, when compared to the 100-volt scale of the simulator, is less than 1%. The search time, depending on the integrator constants, will vary from 0.1 to 2 seconds.

On the basis of the above discussion, we can draw the following conclusions:

* The function V was formed by means of the smooth (thyrite) squaring circuits of the ÉMU-8 simulator. The use of squaring circuits with line-segment approximations was inadmissible, since this produces violent oscillations of the function $\dot{V}$.

1. The problem of searching for the roots of a system of finite equations can be solved by the variable-structure method using an electronic simulator. Every search system can be set up from typical units of standard simulators, for example, the ÉMU-8 simulator or simulators like it which have a differentiation mode.

2. In solving the problem by the variable-structure method, no transformations need be carried out on the functions of the initial system, of the kind, for example, which are necessary when using the well-known gradient method. Usually such transformations lead to a complication of the functions to be set up, as compared with the initial equations, which implies a decrease in accuracy and an increase in the number of computing elements required to set up the scheme.

3. The proposed method should be recommended in particular for the solution of systems of transcendental equations, where the gain in the number of computing elements as compared with the gradient method will be considerable, as well as for the solution of systems of linear algebraic equations. In the solution of linear algebraic equations we no longer have any need for recomputing the initial coefficient matrix [12].

Appendix

We shall prove that the function

$$V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i f_i^2 (x_1, \ldots, x_n) \tag{I}$$

has no relative minima, that is, minima for which $V \neq 0$, if

$$\det A = \frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)} \neq 0$$

everywhere in the region of variation of $(x_1, \ldots, x_n)$.

The minimum points are found among the roots of the system of equations $\partial V/\partial x_i = 0$ $(i = 1, \ldots, n)$. Let us write $\partial V/\partial x_i = 0$ $(i = 1, \ldots, n)$ in expanded form, taking (I) into consideration:

$$\lambda_1 \frac{\partial f_1}{\partial x_1} f_1 + \ldots + \lambda_n \frac{\partial f_n}{\partial x_1} f_n = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lambda_1 \frac{\partial f_1}{\partial x_n} f_1 + \ldots + \lambda_n \frac{\partial f_n}{\partial x_n} f_n = 0.$$

This system of equations can be reduced to a single matrix equation

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{II}$$

or $\lambda A' f = 0$.

Here A' is the transpose of the matrix A, and consequently $\det A' = \det A \neq 0$.

Taking into consideration the fact that $\det \lambda$ is also unequal to zero, we arrive at the conclusion that Eq. (II) has only the trivial solution $f_1 = \ldots = f_n = 0$. Returning to (I), we note that in this case $V = 0$, which proves the proposition.

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A THYRITE MULTIPLIER WITH AN INCREASED PASSBAND

F. B. Gul'ko

(Moscow)
Translated from Avtomatika i Telemekhanika, Vol. 22, No. 12,
pp. 1649-1655, December, 1961
Original article submitted April 4, 1961

A multiplier with thyrite varistors is described which has a passband an order greater then earlier-known multipliers of a similar type. A broadened passband was achieved by compensating for the parasitic component of the thyrite admittances and by replacing the operation of detecting the absolute voltage on the thyrites by that of detecting their absolute currents.

Nonlinear silicon carbide resistors of the thyrite type find rather wide application in analog computing machines and other computers [1-4].

One shortcoming in existing multipliers of this type is their relatively low (up to a few hundred cps) passband. Analyses were made which showed that the passband is chiefly limited by the following factors:

1) the reactive component of the thyrite admittance and the changes of its conductance with an increase in frequency (which is equivalent to the appearance of some parasitic admittance at higher frequencies);

2) the widening of the squared-signal spectrum in the modulus unit located in the signal channel ahead of the thyrite varistors.

By compensating for the parasitic component of the thyrite admittance and by placing the modulus unit after the thyrite varistor the multiplier passband was widened by an order with respect to those previously described [3, 4]. At the same time the static characteristics of the apparatus and the number of operational amplifiers in the arrangement are unchanged.

In order to develop a compensating circuit, thyrite admittance was measured over a range of frequencies up to 20 kc, and its equivalent circuit was established.

I. Investigation of Thyrite Admittance and Synthesis of an Equivalent Circuit

An investigation of thyrite admittance was carried out on type NPS-50-0.7 thyrite varistors over a frequency range from 1 to 20 kc. Measurements were made with a differential transformer circuit (Fig. 1) for an ac voltage $V_{\sim}$ of about 0.5 v on the specimen and a dc bias $V_{=}$, which was varied from 0 to 50 v. A circuit balance was obtained on the fundamental of the ac signal by means of an ac null indicator (NI) for which an ÉO-7 oscillograph was used. The differential transformer T was a ferrocart 2000 toroid having uniformly distributed turns of windings ω_1 (20 turns), ω_2, and ω_3 (over 200 turns). A capacity box was used for the comparison capacity C_0. Due to the lack of a standard resistance box with an operational frequency range higher than 5 kc, type SP variable resistances were used for the comparison resistance R_0, and, after balancing the circuit, their resistance values were measured accurately on a MPT dc bridge. Dc bias on the specimen was set by means of potentiometer P and checked on voltmeter $V_{m=}$. Because of the large size (10 μf) capacitor C_s, the ac signal did not pass through potentiometer P.

If thyrite with a dc bias and a low amplitude ac signal is represented by a parallel equivalent circuit (Fig. 2a), then at the balance point of the circuit the following equations are satisfied

$$R_x = R_0 \frac{w_2}{w_1} \text{ and } C_x = C_0 \frac{w_1}{w_3}.$$

From the results thus obtained the conductance component of the dynamic thyrite admittance can be determined:

$$Y_a = \frac{1}{R_x} = \frac{di_a}{du}$$

and the susceptance component

$$Y_r = \omega C_x = \omega \frac{di_Q}{du} = \omega \frac{i_r\,dt}{du},$$

where i_a and i_r are the in-phase and the reactive currents of the thyrite.

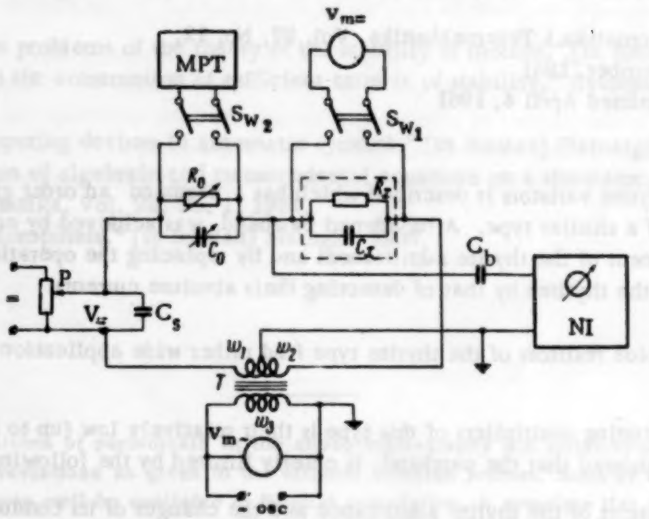

Fig. 1.

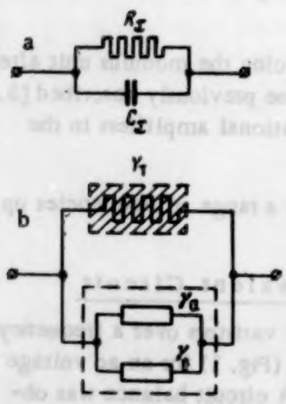

Fig. 2. Equivalent circuits of thyrite. a) Parallel equivalent circuit adopted during the measurements; b) equivalent circuit having a separate parasitic admittance.

From now on it is convenient to separate the parasitic thyrite admittance which is to be compensated. For this purpose thyrite is represented by the equivalent circuit of Fig. 2b, where Y_T is the conductance corresponding to the value of di/du for the thyrite under consideration at dc, and γ_a and γ_r are the conductance and susceptance components, respectively, of the parasitic admittance. Obviously, for every fixed frequency and for every fixed bias voltage

$$\gamma_a = Y_a - Y_T, \quad \gamma_r = Y_r.$$

In Fig. 3 there are shown the values of $\gamma_a = \gamma_a(f)$, and in Fig. 4 the values of $\gamma_r = \gamma_r(f)$ for various bias voltages. An analysis of these curves indicates that an equivalent circuit for thyrite can be represented in the form of a parallel connected resistance R_T, corresponding to the dc resistance of thyrite, a capacity C_∞, and a certain number of RC circuits R_k, C_k having various time constants (Fig. 5a). Capacity C_∞ and resistances R_k apparently depend either little or not at all on voltage, but at least some of the capacities $C_1, C_2, \ldots, C_n$ are nonlinear. At the same time, as indicated by the curves of Fig. 4, this nonlinearity is small for voltages on thyrite up to 10 v; for higher voltages the capacity grows larger for a decrease in frequency. If the equivalent circuit of thyrite is limited to only one RC network R_1C_1, then a compensating circuit can be synthesized in accordance with Fig. 5b. The volt-ampere characteristic of thyrite for frequencies of the order of 5 kc differs appreciably from the static case only for voltages up to 30 v (Fig. 6), i.e., in the portion where the susceptance γ_r does not change significantly with voltage. Therefore C_1 can be a linear capacity in the compensating circuit. An experiment showed that, with a suitable choice of capacities C_∞ and C_1 and of resistance R_1 in the compensating circuit, the resulting volt-ampere characteristic over the frequency range up to 5 kc is practically not any different from the static characteristic of thyrite, whereas without compensation this difference is observed to be worse at frequencies around 1 kc.

II. Circuit of a Thyrite Multiplier with an Increased Passband

In order to achieve the compensation of the parasitic current component for thyrite, in accord with Fig. 5b, there must also be in the circuit, in addition to the squared signals, signals of reverse sign. Another requirement which a multiplier circuit with an increased passband must satisfy is that the voltage on the thyrite varistor must not have a broader frequency spectrum than the spectrum of the squared signals.

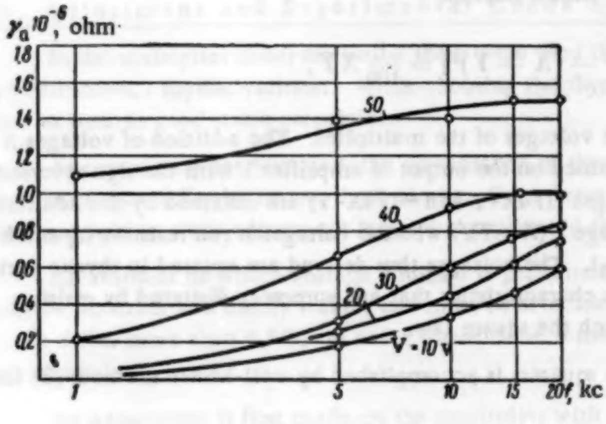

Fig. 3.

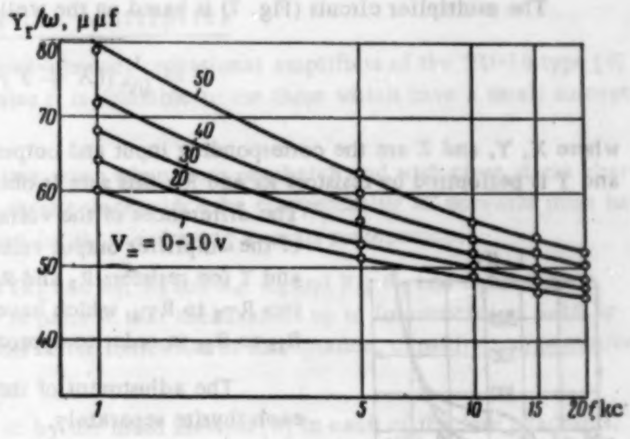

Fig. 4.

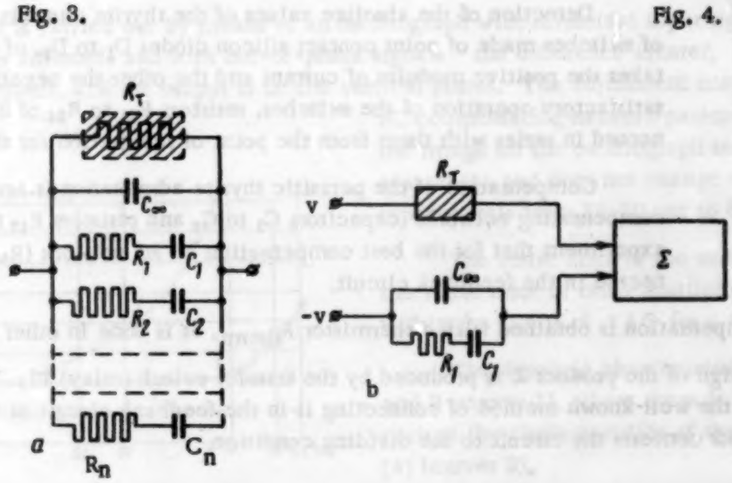

Fig. 5. a) Completely equivalent circuit of a thyrite varistor; b) compensating circuit for the parasitic thyrite admittance.

As was pointed out above, in the previously described multipliers this requirement for signals of changing signs was not fulfilled, since the voltage on the thyrite varistor was equal to the absolute value of the squared signal.

The characteristic of a thyrite varistor* can be described by the expression

$$i_T = au^2 \operatorname{sign} v, \tag{1}$$

where i is the thyrite current, v is the voltage on the thyrite, and a is some positive constant. Upon taking the absolute value of the thyrite voltage,

$$i_T = a\,|v|^2 \operatorname{sign} |v| = a\,|v|^2. \tag{2}$$

If now the absolute value of the thyrite current is taken, then (1) can be written in the form

$$i_T = |av^2 \operatorname{sign} v| = av^2. \tag{3}$$

* It is assumed that the characteristic of a thyrite varistor is made closely quadratic by well-known methods [3].

From a comparison of (2) and (3) it is obvious that for accurate multiplier operation it is necessary in the first case to ensure for thyrite varistors fulfilling relation (1) that there is a significantly wider frequency range than in the second case. Therefore in the circuit that was developed the absolute value of the thyrite current was detected and not the absolute value of its voltage.

The multiplier circuit (Fig. 7) is based on the well-known relation

$$Z = \frac{1}{400}[(X+Y)^2 - (X-Y)^2] = \frac{1}{100}XY,$$

where X, Y, and Z are the corresponding input and output voltages of the multiplier. The addition of voltages X and Y is performed by resistors R_3 and R_4; this sum is obtained on the output of amplifier 1 with the sign reversed. The differences of the voltages (1/4X-Y and −1/4X-Y) are obtained by the addition of the amplifier output voltage I-(X + Y)/2 withthe voltages X (on resistors R_5 and R_6) and Y (on resistors R_7 and R_8). The voltages thus derived are squared in thyrite varistors R_{T1} to R_{T4}, which have characteristics that are purposely distorted by resistors R_{13} to R_{16} in order to approach the square law.

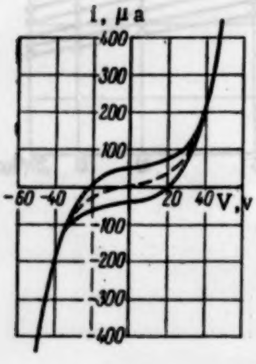

Fig. 6.

The adjustment of the squarers is accomplished by well-known methods [3] for each thyrite separately.

Detection of the absolute values of the thyrite currents is performed by two pairs of switches made of point contact silicon diodes D_1 to D_8, of which one pair of switches takes the positive modulus of current and the other the negative. In order to ensure satisfactory operation of the switches, resistors R_{21} to R_{24},of 3.6 kohm in value,are connected in series with them from the point of summation for the output amplifier 2.

Compensation of the parasitic thyrite admittance is accomplished by four RC compensating networks (capacitors C_1 to C_8 and resistors R_{17} to R_{20}). It was found by experiment that for the best compensation an RC network (R_{28}, C_9) should also be connected in the feedback circuit.

Temperature compensation is obtained with a thermistor R_{therm}, as is done in other types of multipliers.

A change in the sign of the product Z is produced by the transfer switch (relay) R1. The multiplier can be made into a divider by the well-known method of connecting it in the feedback circuit of the output amplifier. Transfer switch (relay) R2 converts the circuit to the dividing condition.

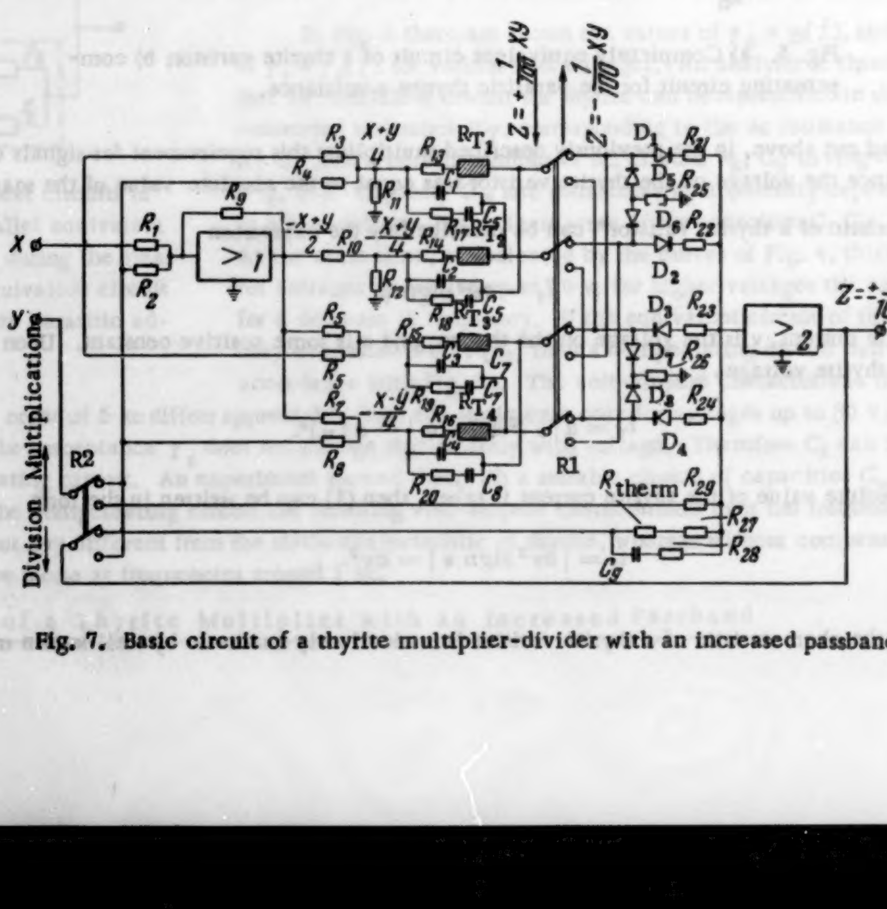

Fig. 7. Basic circuit of a thyrite multiplier-divider with an increased passband.

In the completed model of the multiplier an automatic arrangement was proved for the dividing condition to ensure circuit stability by making the sign of the multiplier output current such that the feedback was always negative. This was achieved with a simple circuit (not shown in Fig. 7),which reversed the sign by means of relay Rl when the sign of the dividend (voltage X) changed.

III. Adjustment and Experimental Check of the Multiplier

In the multiplier under discussion there were used three-channel operational amplifiers of the TU-10 type [5] and NSP-50-0.7 thyrite varistors. When choosing the thyrites it is desirable to use those which have a small susceptance as well as good static properties.

As experience has revealed, the susceptance of thyrites, even when from one batch and with close static characteristics, can differ by two or three to one. For a large thyrite susceptance the compensating RC networks must have large capacities; their adjustment is more complicated than for thyrites having a small susceptance.

All resistors on which voltage addition is performed (R_1 and R_2; R_3 and R_4; R_5 and R_6; R_7 and R_8 in Fig. 7) must be accurate and highly stable, and must be selected in pairs so that their values up to frequencies of 5-10 kc do not differ more than 0.5%. In cases where these resistors differ somewhat in susceptance, capacity compensation can be used.

An adjustment is first made on the multiplier with dc by the usual method [3] in each of the four quadrants.

The ac adjustment is carried out by means of an oscillograph with sinusoidal input signals. With in-phase signals the sum squarer is adjusted, and with out-of-phase signals – the difference squarer. One of the input signals is fed to the horizontal plates, and the output is on the vertical plates. The adjustment consists of selecting the RC compensating network parameters in such a way that the image on the oscillograph screen takes the shape of a parabola and does not change with a variation of the input signals from 30-50 cps to 5 kc.

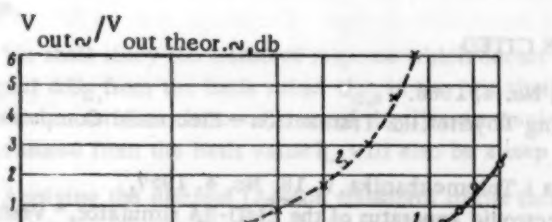

Fig. 8.

The static error of the multiplier as described, like the static error of other multipliers of similar type, lies within the limits of ± 1% for a maximum value of 100 v.

The dynamic characteristics are shown in Figs. 8 and 9 (curves 1), where there is also introduced for comparison the characteristics of the multiplier described in [4] (curves 2).

Figure 8 shows the relation of the divergence in the effective ac component at the output, when two sinusoidal signals at an amplitude of 100 v are being multiplied, from the theoretical value of this magnitude as a function of the input signal frequency.

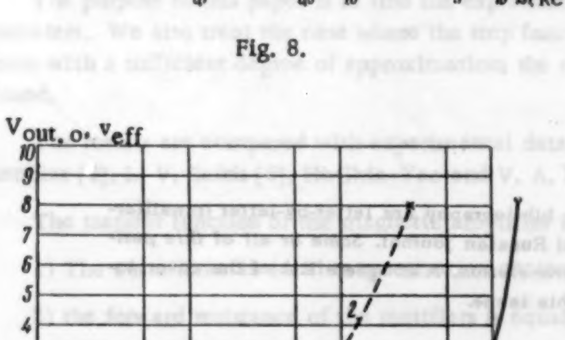

Fig. 9. Frequency dependence of the error for the multiplication of a sinusoidal signal X = 100 sin $2\pi f t$ (v) by zero.

As the curves indicate, an appreciably divergence of the output signal from the theoretical begins for the multiplier under discussion only at frequencies around 5 kc (curve 1), while for the multiplier of [4] such a divergence starts at 300 cps.

The results of multiplying a sinusoidal signal having an amplitude of 100 v by zero is presented in Fig. 9 (the frequency of the input signal is plotted along the axis of abscissas). In this case the error begins to increase only for frequencies around 5 kc (curve 1) but for the multiplier of [4] it is around 500 cps.

It should be noted that the characteristics presented correspond to multiplier operation under the most difficult conditions when there are sign-changing signals on the squares. The method used earlier for the determination of frequency characteristics when multiplying a sinusoidal signal by a dc signal of large value cannot completely

characterize a multiplier since in this case the signals on the squarers are sign-invariant, the modulus detector unit is not operating, and consequently it does not introduce any errors. The condition for the multiplier is thereby made rather easy.

The dc component of the output signal due to the multiplication of two sinusoidal signals does not change up to frequencies of the order of 15 kc.

CONCLUSIONS

1. The passband of thyrite multipliers is limited by the parasitic component of the thyrite admittances for high frequencies and by the broadening of the spectrum of the squared signals during the detection of their moduli.

2. The frequency characteristics of thyrite varistors were studied and an equivalent substitute circuit was obtained which was used as the basis for the development of a method of compensation for the parasitic component of thyrite admittance.

3. A thyrite squaring circuit was developed which detects the modulus of the thyrite current, and in which no broadening of the squared-signal spectrum occurs.

4. A multiplier was constructed using thyrites with frequency compensation and a current modulus detector circuit which has a passband of about 5 kc with sign-varying signals on both inputs.

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THE TRANSFER FUNCTION OF A SELF-SATURATING MAGNETIC AMPLIFIER WITH A dc RESISTIVE-INDUCTIVE LOAD FOR A STEP INPUT SIGNAL

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Original article submitted March 9, 1961

The paper derives the expression for the transfer function of an ideal self-saturating magnetic amplifier with a dc resistive-inductive load in the case where the time constant of the load and the power gain are large. The results are compared with those of other authors and with other experimental results.

Statement of the Problem

We shall assume that the input and output signals of a magnetic amplifier with a dc resistive-inductive load (Fig. 1) are the average values of the control voltage U_C and the load current I_L over a half-cycle of the supply voltage.

We shall study the transient response which occurs in the magnetic amplifier for a small deviation of the input signal ΔU_C from the basis value $U_{C,0}$ in the initial stationary mode. We shall assume that ΔU_C is a step time function that satisfies the conditions of the discrete Laplace transform [1]. Then the increment in the output signal ΔI_L, measured from the basis value I_{L0} will also be a step time function (Fig. 2).

Applying the discrete Laplace transform to the increments of the input and output signals, we obtain the transfer function for the magnetic amplifier; this function is determined by the ratio between the output signal transform and the input signal transform.

The purpose of this paper is to find the expression for the transfer function in terms of the magnetic amplifier parameters. We also treat the case where the step functions for the input and output signals can be treated as continuous with a sufficient degree of approximation; the expression for the transfer function corresponding to this case is found.

The results are compared with experimental data and with the conclusions drawn by H. F. Storm [2, 3], M. A. Rozenblat [4], L. V. Safris [5], HuChia-Yao and V. A. Shubenko [6].

The transfer function of the magnetic amplifier is determined for the following assumptions:

1) The cores have an ideal magnetization characteristic (Fig. 3);

2) the forward resistance of the rectifiers is equal to zero, and the reverse resistance is equal to infinity;

3) there is no magnetic leakage;

4) the resistance of the operating windings w_0 of the magnetic amplifier is equal to zero.

Certain additional limitations will be added below.

The Equations for the Magnetic Amplifier Circuits

It is convenient to formulate the equations for the magntic amplifier circuits in terms of the normalized quantities

$$U'_m = \frac{U_m}{w_o}, \quad U'_{c.} = \frac{U_{c.}}{w_{c.}}, \quad i'_{c.} = i_c w_{c.}, \quad i'_o = i_o w_o,$$

$$i'_L = i_L w_o, \quad R' = \frac{R}{w_o^2}, \quad L' = \frac{L}{w_o^2}, \quad r' = \frac{r}{w_{c.}^2}. \tag{1}$$

Here U_m is the amplitude value of the supply voltage which varies according to the law $u = U_m \sin \omega_0 t$; $U_{c.}$ is the instantaneous value of the control voltage; $i_{c.}, i_o, i_L$ are the instantaneous values for the control current, the current in the operating windings w_o, and the load current; R and L are the resistance and inductance of the load; r is the resistance of the control circuit; $w_{c.}$ and w_o are the number of turns on the control winding and the operating winding.

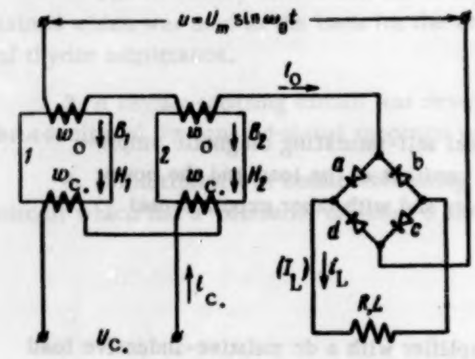

Fig. 1. Circuit for the self-saturating magnetic amplifier.

The primes denote the corresponding normalized quantities.

We shall use B_1, B_2, H_1 and H_2 to denote the instantaneous values of the flux densities and field intensities in cores 1 and 2 (Fig. 1).

The currents in the windings are related to the intensities in the cores by the equations

$$i'_o = \frac{l}{2}(H_1 + H_2), \quad i'_{c.} = \frac{l}{2}(H_1 - H_2), \tag{2}$$

where l is the length of the median line of force in the core.

The equations for the magnetic amplifier circuit depend on the state of the rectifier bridge. The bridge may be in three states: positive admittance (the rectifiers a and c conduct), negative admittance (rectifiers b and d conduct) and short-circuited (all four rectifiers conduct).

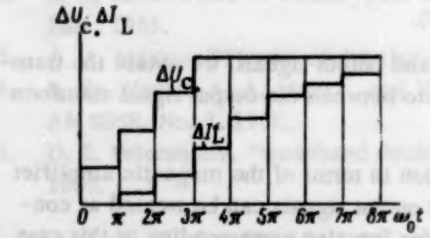

Fig. 2. Step functions for the input and output signals.

For the positive and negative admittance states the equations of the circuits for instantaneous values of the currents are written as

$$U'_m \sin \omega_0 t = i'_o R + L' \frac{di'_o}{dt} + S\left(\frac{dB_1}{dt} + \frac{dB_2}{dt}\right), \tag{3}$$

$$U'_{c.} = i'_{c.} r' + S\left(\frac{dB_1}{dt} - \frac{dB_2}{dt}\right), \tag{4}$$

$$i'_o = i'_L \tag{5}$$

(for positive admittance),

$$i'_o = -i'_L \tag{6}$$

(for negative admittance).

Here S is the core cross section.

For a short-circuited state of the rectifier bridge the equations for the circuits are written as

$$U'_m \sin \omega_0 t = S\left(\frac{dB_1}{dt} + \frac{dB_2}{dt}\right), \tag{7}$$

$$U'_{c.} = i'_{c.} r' + S\left(\frac{dB_1}{dt} - \frac{dB_2}{dt}\right), \tag{8}$$

$$i'_L R' + L' \frac{di'_L}{dt} = 0. \tag{9}$$

Within the limits of a half-cycle of line voltage it is possible to isolate four intervals which are characterized by the fact that within each of them the state of the rectifier bridge and the magnetic state of the cores (saturated or unsaturated) remain the same.

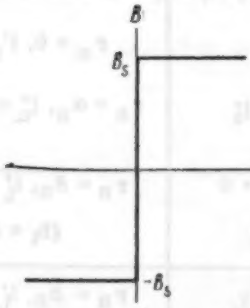

Fig. 3. Magnetization characteristic for an ideal core.

Figure 4 shows the graphs of u', i'_0, i'_L, $i'_{c.}$, B_1 and B_2 for two adjacent half-cycles of line voltage denoted by the subscripts n and n + 1; here it is assumed that the subscript n correspond to the positive half-cycle of line voltage. The graphs were plotted for a mode with free current harmonics in the control circuit; this mode is characteristic for magnetic amplifiers with $K_0 > 1$ (K_0 is the power gain). The graphs show the phase angles α_n, β_n and γ_n which correspond to the boundaries of the intervals in the n-th half-cycle, as well as the limiting values i'_{0n}, $i'_{\beta n}$ and $i'_{\gamma n}$ for the load current. In studying the processes within the half-cycle n we introduce the independent variable τ_n instead of $\omega_0 t$; this variable is measured from the initial point of the n-th half-cycle. Analogous notation, which differs only in the subscripts, is used for the subsequent intervals.

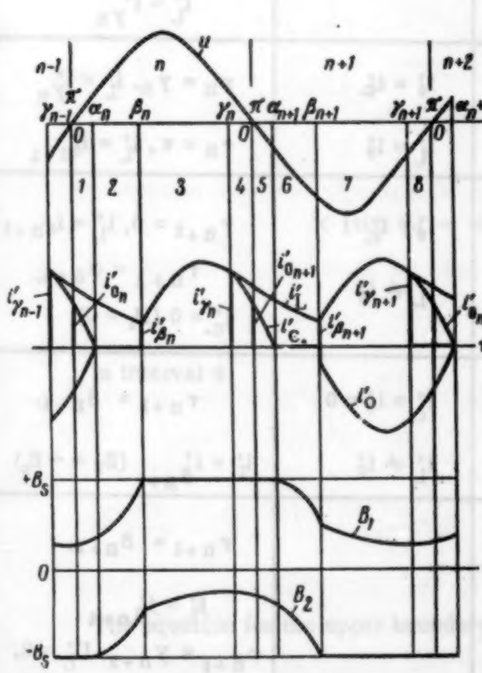

Fig. 4. Graphs for the instantaneous values of u', i'_0, i'_L, i'_c, B_1 and B_2.

The table shows the states of the rectifier bridge and the cores for each of the intervals in the half-cycles n and n + 1, as well as the special conditions within the intervals n and their boundaries.

Within each of the intervals the processes in the magnetic amplifier are described by linear differential equations with constant coefficients; the values of the coefficients depend on the number of the interval. Taking the data and the table into account, we find the solution for Eqs. (2)-(9) in each of the intervals for the half-cycles n and n + 1.

Within interval 1

$$i'_L = i'_{0_n} e^{-\frac{\tau_n}{\omega_0 T_L}}, \tag{10}$$

where

$$T_L = \frac{L'}{R'} \tag{11}$$

is the time constant of the load.

The equation for the upper boundary of interval 1 is

$$U'_m \sin \alpha_n = U'_{c.,n} \tag{12}$$

Here $U'_{c.n}$ is the normalized value of the control voltage in the interval n.

During the interval 2

$$i'_L = i'_{0_n} e^{-\frac{\tau_n}{\omega_0 T_L}},$$

$$S\frac{dB_2}{dt} = \frac{1}{2}\left(U'_m \sin \tau_n - U'_{c.,n}\right). \tag{13}$$

The equation for the upper boundary of interval 2 is

$$i'_{\beta_n} = i'_{0_n} e^{-\frac{\beta_n}{\omega_0 T_L}}. \tag{14}$$

Number of intervals	State of the cores 1	State of the cores 2	State of the rectifier bridge	B and H	Valid equations	The relationship between the currents	Boundary conditions
1	us	s	sc	$H_1 = 0$ $B_2 = -B_s$	(2), (7)-(9)	$i'_{C.} = i'_0$ $i'_L \neq -i'_0$	$\tau_n = 0,\ i'_L = i'_{0n}$ $\tau_n = \alpha_n,\ i'_{C.} = 0\ (H_2 = 0)$
2	us	us	sc	$H_1 = 0$ $H_2 = 0$	(2), (7)-(9)	$i'_{C.} = i'_0 = 0$ $i'_L \neq i'_0$	$\tau_n = \beta_n,\ i'_L = i'_{\beta n}$ $(B_1 = B_s)$
3	s	us	+	$H_2 = 0$ $B_1 = B_s$	(2), (3)-(5)	$i'_0 = i'_L = i'_{C.}$	$\tau_n = \beta_n,\ i'_L = i'_{\beta n}$ $\tau_n = \gamma_n,\ U'_L = 0,$ $i'_L = i'_{\gamma n}$
4	s	us	sc	$H_2 = 0$ $B_1 = B_s$	(2), (7)-(9)	$i'_0 = i'_{C.}$ $i'_L = i'_0$	$\tau_n = \gamma_n,\ i'_L = i'_{\gamma n}$ $\tau_n = \pi,\ i'_L = i'_{0n+1}$
5	s	us	sc	$H_2 = 0$ $B_1 = B_s$	(2), (7)-(9)	$i'_0 = i'_{C.}$ $i'_L \neq i'_0$	$\tau_{n+1} = 0,\ i'_L = i_{0n+1}$ $\tau_{n+1} = \alpha_{n+1},$ $i'_{C.} = 0\ (H_1 = 0)$
6	us	us	sc	$H_1 = 0$ $H_2 = 0$	(2), (7)-(9)	$i'_{C.} = i'_0 = 0$ $i'_L \neq i'_0$	$\tau_{n+1} = \beta_{n+1},$ $i'_L = i'_{\beta n+1}\ (B_2 = -B_s)$
7	us	s	−	$H_1 = 0$ $B_2 = -B_s$	(2)-(4), (6)	$i'_L = i'_{C.} = -i'_0$	$\tau_{n+1} = \beta_{n+1},$ $i'_L = i'_{\beta n+1}$ $\tau_{n+1} = \gamma_{n+1},\ U'_L = 0,$ $i'_L = i'_{\gamma n+1}$
8	us	s	sc	$H_1 = 0$ $B_2 = -B_s$	(2), (7)-(9)	$i'_0 = -i'_{C.}$ $i'_L \neq -i'_0$	$\tau_{n+1} = \gamma_{n+1}$ $i'_L = i'_{\gamma n+1}$ $\tau_{n+1} = \pi,\ i'_L = i'_{0n+2}$

Note: In the table s denotes a saturated core state; us denotes an unsaturated core state; sc denotes the short-circuited state for the bridge; + and − denote the states of positive and negative admittance for the bridge.

In interval 3

$$i'_L = \frac{U'_m}{(R'+r')[1+(\omega_0 T)^2]}\left[\sin\tau_n - \omega_0 T\cos\tau_n - e^{-\frac{\tau_n-\beta_n}{\omega_0 T}}(\sin\beta_n - \omega_0 T\cos\beta_n)\right] + \frac{U'_{C_n}}{R'+r'}\left(1-e^{-\frac{\tau_n-\beta_n}{\omega_0 T}}\right) + i'_{\beta_n} e^{-\frac{\tau_n-\beta_n}{\omega_0 T}}, \tag{15}$$

where

$$T = \frac{L'}{R'+r'}, \tag{16}$$

$$S\frac{dB_2}{dt} = \frac{r'}{R'+r'}U'_m\sin\tau_n - \frac{R'}{R'+r'}U'_{C_n} - Tr'\frac{di'_L}{dt}. \tag{17}$$

The equations for the upper boundary of the interval 3 are

$$i'_{\gamma_n} = \frac{1}{r'}(U'_m\sin\gamma_n + U'_{C_n}), \tag{18}$$

$$i'_{\gamma_n} = \frac{U'_m}{(R'+r')[1+(\omega_0 T)^2]} \times \left[\sin\gamma_n - \omega_0 T\cos\gamma_n - e^{-\frac{\gamma_n-\beta_n}{\omega_0 T}}(\sin\beta_n - \omega_0 T\cos\beta_n)\right] + \frac{U_{C_n}}{R'+r'}\left(1-e^{-\frac{\gamma_n-\beta_n}{\omega_0 T}}\right) + i'_{\beta_n} e^{-\frac{\gamma_n-\beta_n}{\omega_0 T}}. \tag{19}$$

In interval 4

$$i'_L = i'_{\gamma_n} e^{-\frac{\tau_n-\gamma_n}{\omega_0 T_L}}, \tag{20}$$

$$S\frac{dB_2}{dt} = U'_m\sin\tau_n. \tag{21}$$

The equation for the upper boundary of interval 4 is

$$i'_{\alpha_{n+1}} = i'_{\gamma_n} e^{-\frac{\pi-\gamma_n}{\omega_0 T_L}}. \tag{22}$$

The current i'_L during the half-cycle n + 1 is expressed in terms of the corresponding intervals, using formulas (10), (15) and (20) under conditions where the subscripts n are replaced by n + 1; during the half-cycle n + 2 the subscripts n are replaced by n + 2. The same applies to the boundary equations (12), (14), (18), (19) and (22).

In interval 5

$$S\frac{dB_2}{dt} = -U'_m\sin\tau_{n+1}. \tag{23}$$

In interval 6

$$S\frac{dB_2}{dt} = -\frac{1}{2}(U'_m\sin\tau_{n+1} + U'_{C_{n+1}}). \tag{24}$$

We shall express the average value of the load current during the half-cycle n by

$$I'_{L_n} = \frac{1}{\pi}\int_0^{\pi} i'_L d\tau_n. \tag{25}$$

We shall substitute the value of i'_L for the intervals 1-4 into (25) from (10), (11), (15), and (20). Taking (14), (18), (19) into account, we obtain

$$\begin{aligned}
\pi r' I'_{L_n} = U'_m \Bigg\{ \Bigg[\sin\gamma_n e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} \\
- \frac{r'}{R'+r'} \frac{(\sin\gamma_n - \omega_0 T\cos\gamma_n)\, e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} - \sin\beta_n + \omega_0 T\cos\beta_n}{1+(\omega_0 T)^2} \Bigg] \\
\times \left[\omega_0 T_L \left(e^{\frac{\beta_n}{\omega_0 T_L}} - 1\right) + \omega_0 T\right] + \omega_0 T_L \sin\gamma_n \left(1 - e^{-\frac{\pi-\gamma_n}{\omega_0 T_L}}\right) \\
+ \frac{r'}{R'+r'} (\cos\beta_n - \cos\gamma_n) - \omega_0 T\sin\gamma_n \Bigg\} + \\
+ U'_{C_n} \Bigg\{ \left[e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} - \frac{r'}{R'+r'}\left(e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} - 1\right)\right] \left[\omega_0 T_L \left(e^{\frac{\beta_n}{\omega_0 T_L}} - 1\right) + \omega_0 T\right] \\
+ \omega_0 T_L \left(1 - e^{-\frac{\pi-\gamma_n}{\omega_0 T_L}}\right) + \frac{r'}{R'+r'}(\gamma_n - \beta_n) - \omega_0 T \Bigg\}.
\end{aligned} \tag{26}$$

For the half-cycles n + 1 and n + 2 we obtain analogous equations (26a) and (26b) when n is replaced by n + 1 and n + 2, respectively.

At the end of the first and fifth intervals the flux density in the second core has the identical value $B_2 = -B_S$. Therefore

$$\int_{(B_2)\alpha_n}^{(B_2)\beta_{n+1}} dB_2 = 0$$

or

$$S\int_{\alpha_n}^{\pi} \frac{dB_2}{dt} d\tau_n + S\int_0^{\beta_{n+1}} \frac{dB_2}{dt} d\tau_{n+1} = 0. \tag{27}$$

We shall substitute $S(dB_2/dt)$ into (27) from (13), (17), (21), (23) and (24), and taking (18) and (19) into account we shall obtain

$$\begin{aligned}
U'_m \Bigg\{ \frac{\cos\alpha_n + \cos\alpha_{n+1} + \cos\beta_{n+1}}{2} - \frac{R'-r'}{2(R'+r')}\cos\beta_n + \frac{R'}{R'+r'}\cos\gamma_n \\
+ \omega_0 T \Bigg[\left(e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} - 1\right)\sin\gamma_n \\
- \frac{r'}{R'+r'} \frac{(\sin\gamma_n - \omega_0 T\cos\gamma_n)\, e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} - \sin\beta_n + \omega_0 T\cos\beta_n}{1+(\omega_0 T)^2} \Bigg] \Bigg\} \\
+ U'_{C_n} \left\{ \frac{\alpha_n}{2} + \frac{R'-r'}{2(R'+r')}\beta_n - \frac{R'}{R'+r'}\gamma_n + \omega_0 T \frac{R'}{R'+r'}\left(e^{\frac{\gamma_n-\beta_n}{\omega_0 T}} - 1\right)\right\} \\
+ U'_{C_{n+1}} \frac{\alpha_{n+1} - \beta_{n+1}}{2} = 0.
\end{aligned} \tag{28}$$

The analogous Eq. (28a) is obtained for the half-cycles n + 1 and n + 2 when n is replaced by n + 1 and n + 1 is replaced by n + 2 in (28).

We shall express $i'_{0_{n+1}}$ in terms of the boundary phase angles for the intervals during the half-cycles n and n + 1.

By analogy with (14), (18) and (19) we shall write

$$i'_{\beta_{n+1}} = i'_{0_{n+1}} e^{-\frac{\beta_{n+1}}{\omega_0 T_L}}, \quad i'_{\gamma_{n+1}} = \frac{U'_m \sin\gamma_{n+1} + U'_{\gamma_{n+1}}}{r'},$$

$$i'_{\gamma_{n+1}} = \frac{U'_m}{(R'+r')\,[1+(\omega_0 T)^2]}$$

$$\times \left[\sin\gamma_{n+1} - \omega_0 T\cos\gamma_{n+1} - e^{-\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}} (\sin\beta_{n+1} - \omega_0 T\cos\beta_{n+1})\right] \tag{29}$$

$$+ \frac{U'_{C_{n+1}}}{R'+r'}\left(1 - e^{-\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}}\right) + i'_{\beta_{n+1}} e^{-\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}}$$

for the (n + 1)-th half-cycle.

Eliminating $i'_{0_{n+1}}$, i_{γ_n}, $i_{\gamma_{n+1}}$ and $i_{\beta_{n+1}}$ from (18), (22) and (29), we obtain the equation

$$U'_m \left\{\sin\gamma_n e^{-\frac{\pi-\gamma_n+\beta_{n+1}}{\omega_0 T_\pi}} - \sin\gamma_{n+1} e^{\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}} + \frac{r'}{R'+r'}\,\frac{1}{1+(\omega_0 T)^2}\right.$$

$$\left.\times\left[(\sin\gamma_{n+1} - \omega_0 T\cos\gamma_{n+1})\, e^{\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}} - \sin\beta_{n+1} + \omega_0 T\cos\beta_{n+1}\right]\right\} \tag{30}$$

$$= U'_{C,n+1}\left\{e^{\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}} - \frac{r'}{R'+r'}\left(e^{\frac{\gamma_{n+1}-\beta_{n+1}}{\omega_0 T}} - 1\right)\right\} - U'_{C,n} e^{-\frac{\pi-\gamma_n+\beta_{n+1}}{\omega_0 T_L}}.$$

When n is replaced by n + 1 and n + 1 is replaced by n + 2 we obtain the analogous Eq. (30a) for the half-cycles n + 1 and n + 2.

The system of ten equations (12), (12a), (12b), (26), (26a), (26b), (28), (28a), (30) and (30a) related the nine unknowns $\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \beta_n, \beta_{n+1}, \beta_{n+2}, \gamma_n, \gamma_{n+1}, \gamma_{n+2}$ to the sought-for step function I'_L and the known step function $U'_{C,}$.

The System of Equations for the Magnetic Amplifier in the Case where $\omega_0 T_L >> 1$ and $r' << R'$

In the case under study where the time constant of the load and the power gain are large the system of equations interrelating I'_L, U'_C and the expressions for the boundary angles of the intervals are appreciably simplified.

We shall use the substitutions

$$I'_{Lm} = \frac{2}{\pi}\,\frac{U'_m}{R'}, \tag{31}$$

$$I'_{C_n st} = \frac{U'_{C_n}}{r'}. \tag{32}$$

Here I'_{Lm} is the maximum load current; $I'_{C,nst}$ is the stationary control current corresponding to $U'_{C,n}$.

In accordance with (12), (31) and (32),

$$\sin\alpha_n = \frac{U'_{C,n}}{U'_m} = \frac{2}{\pi}\,\frac{I'_{C,nst}}{I'_{Lm}}\,\frac{r'}{R'}. \tag{33}$$

In the working zone the input-output characteristic is $I'_{C,nst}/I'_{Lm} < 1$. For $r'/R' \to 0$, $\alpha_n \to 0$. Therefore,

$$\sin \alpha_n \approx \alpha_n, \quad \cos \alpha_n \approx 1. \tag{34}$$

In the analysis below we shall demonstrate that for $r'/R' \to 0$, $\gamma_n \to \pi$.

We shall assume

$$\gamma_n = \pi - \theta_n, \tag{35}$$

where $\theta_n \ll \pi$.

Then

$$\sin \gamma_n = \theta_n, \quad \cos \gamma_n = -1. \tag{36}$$

We shall use the expansion of exponents into a power series with the retention of only the first two terms. Taking (34) and (36) into account, we shall transform the system of equations (12), (26), (28) and (30). In the transformed equations we shall retain only the terms with an order of smallness no higher than the first (small quantities of the first order are α, θ, $\frac{r'}{R'}\frac{1}{\omega_0 T_L}$). We shall also replace T by T_L and $(\omega_0 T)^2 + 1$ by $(\omega_0 T_L)^2$. Then

$$\alpha_n = \frac{U'_{c,n}}{U'_m}, \tag{37}$$

$$\frac{r'}{U'_m} I'_{L_n} = \theta_n + \alpha_n, \tag{38}$$

$$\frac{\cos \beta_{n+1} - \cos \beta_n}{2} + (\pi - \beta_n)\,\theta_n = \frac{\alpha_n \beta_n}{2} + \frac{\alpha_{n+1}\beta_{n+1}}{2}, \tag{39}$$

$$\begin{aligned}(\omega_0 T_L - \beta_{n+1})\,\theta_n - (\omega_0 T_L + \pi - \beta_{n+1})\,\theta_{n+1} + \frac{r'}{R'}(1 + \cos \beta_{n+1}) \\ = \alpha_{n+1}(\omega_0 T_L + \pi - \beta_{n+1}) - \alpha_n(\omega_0 T_L - \beta_{n+1}).\end{aligned} \tag{40}$$

From (37)-(40) we shall obtain the following equations for the stationary mode when $\theta_n = \theta_{n+1} = \theta$ and etc.:

$$\alpha = \frac{U'_c}{U'_m}, \tag{41}$$

$$\frac{r'}{U'_m} I'_L = \theta + \alpha, \tag{42}$$

$$\theta = \alpha \frac{\beta}{\pi - \beta}, \tag{43}$$

$$\frac{r'}{R'} = \frac{\pi(\alpha + \theta)}{1 + \cos \beta}. \tag{44}$$

From (41)-(44) we find

$$I'_L = I'_{Lm} \frac{1 + \cos \beta}{2}, \tag{45}$$

$$I'_c = \frac{U'_y}{r'} = I'_L \frac{\pi - \beta}{\pi}, \tag{46}$$

where the maximum load current I'_{Lm} is expressed by formula (31).

The smallness of θ for small α and r'/R' can be judged in accordance with (44).

Equations (39) and (40) are nonlinear. After subjecting (37)-(40) to linearization, we obtain a system of equations for the relationships between the small increments in the sought-for quantities:

$$\Delta\alpha_n = \frac{\Delta U'_{C,n}}{U'_m}, \tag{47}$$

$$\frac{r'}{U'_m}\Delta I'_{L,n} = \Delta\theta_n + \Delta\alpha_n, \tag{48}$$

$$\left(\frac{\sin\beta-\alpha}{2}-\theta\right)\Delta\beta_n - \frac{\sin\beta+\alpha}{2}\Delta\beta_{n+1} + (\pi-\beta)\Delta\theta_n = \frac{\beta}{2}(\Delta\alpha_n + \Delta\alpha_{n+1}), \tag{49}$$

$$\begin{aligned}(\omega_0 T_L - \beta)\,\Delta\theta_n - (\omega_0 T_L + \pi - \beta)\,\Delta\theta_{n+1} - \frac{r'}{R'}\sin\beta\,\Delta\beta_{n+1} \\ = (\omega_0 T_L + \pi - \beta)\,\Delta\alpha_{n+1} - (\omega_0 T_L - \beta)\,\Delta\alpha_n.\end{aligned} \tag{50}$$

By a rotating transposition of the subscripts in (47)-(50) we obtain analogous equations for the subsequent intervals (47a)-(50a).

$$\Delta\alpha_{n+1} = \frac{\Delta U'_{C,n+1}}{U'_m}, \tag{47a}$$

$$\Delta\alpha_{n+2} = \frac{\Delta U'_{C,n+2}}{U'_m}, \tag{47b}$$

$$\frac{r'}{U'_m}\Delta I'_{L_{n+1}} = \Delta\theta_{n+1} + \Delta\alpha_{n+1}, \tag{48a}$$

$$\frac{r'}{U'_m}\Delta I'_{L_{n+2}} = \Delta\theta_{n+2} + \Delta\alpha_{n+2}, \tag{48b}$$

$$\begin{aligned}\left(\frac{\sin\beta-\alpha}{2}-\theta\right)\Delta\beta_{n+1} - \frac{\sin\beta+\alpha}{2}\Delta\beta_{n+2} + (\pi-\beta)\Delta\theta_{n+1} \\ = \frac{\beta}{2}(\Delta\alpha_{n+1} + \Delta\alpha_{n+2}),\end{aligned} \tag{49a}$$

$$\begin{aligned}(\omega_0 T_L - \beta)\,\Delta\theta_{n+1} - (\omega_0 T_L + \pi - \beta)\,\Delta\theta_{n+2} - \frac{r'}{R'}\sin\beta\,\Delta\beta_{n+2} \\ = (\omega_0 T_L + \pi - \beta)\,\Delta\alpha_{n+2} - (\omega_0 T_L - \beta)\,\Delta\alpha_{n+1}.\end{aligned} \tag{50a}$$

The ten equations (47), (47a), (47b), (48), (48a), (48b), (49), (49a), (50) and (51a) interrelate the unknown step function I'_L and the increments in the boundary angles $\Delta\alpha_n$, $\Delta\alpha_{n+1}$, $\Delta\alpha_{n+2}$, $\Delta\beta_n$, $\Delta\beta_{n+1}$, $\Delta\beta_{n+2}$, $\Delta\theta_n$, $\Delta\theta_{n+1}$, $\Delta\theta_{n+2}$ with the known step function $\Delta U'_C$.

Eliminating the intermediate variables, we obtain a linear equation in finite differences for the step function $\Delta I'_L$.

$$A\,\Delta I'_{L_n} - A_1\Delta I'_{L,n+1} + A_2\Delta I'_{L_{n+2}} = B_1\Delta U'_{C,n+1} + B_2\Delta U'_{C,n+2}. \tag{51}$$

Here

$$\begin{aligned}A &= (\omega_0 T_L - \beta)\left(\frac{\sin\beta-\alpha}{2}-\theta\right);\\ A_1 &= (\omega_0 T_L - \beta)(\sin\beta-\theta) + \left(\frac{\sin\beta-\alpha}{2}-\theta\right)\pi - \frac{r'}{R'}(\pi-\beta)\sin\beta,\\ A_2 &= (\omega_0 T_L + \pi - \beta)\frac{\sin\beta+\alpha}{2},\quad B_1 = \frac{1}{R'}\left(\pi-\frac{\beta}{2}\right)\sin\beta,\quad B_2 = \frac{1}{R'}\frac{\beta}{2}\sin\beta.\end{aligned} \tag{52}$$

The Transfer Function for the Magnetic Amplifier

We shall denote the transforms (in the sense of a discrete Laplace transform) of the step functions $\Delta I'_L$ and $\Delta U'_C$ by $\Delta I'_L(q)$ and $\Delta U'_C(q)$, respectively (here q is a dimensionless operator).

The transfer function of the magnetic amplifier is defined as

$$Y(q) = \frac{\Delta I'_L(q)}{\Delta U'_C(q)}. \tag{53}$$

Assuming that in the initial basis mode the increments of the functions and their first differences are equal to zero, we shall apply a discrete Laplace transform to (51) and obtain the transfer function for the magnetic amplifier in accordance with (53):

$$Y(q) = \frac{B_1 e^q + B_2 e^{2q}}{A - A_1 e^q + A_2 e^{2q}}. \tag{54}$$

Expression (54) can be transformed to

$$Y(q) = K \frac{1 + \tau_a (e^q - 1) + \tau_b^2 (e^q - 1)^2}{1 + \tau_c (e^q - 1) + \tau_d^2 (e^q - 1)^2}, \tag{55}$$

where

$$K = \frac{B_1 + B_2}{A - A_1 + A_2}, \quad \tau_a = \frac{B_1 + 2B_2}{B_1 + B_2}, \quad \tau_b^2 = \frac{B_2}{B_1 + B_2},$$
$$\tau_c = \frac{2A_2 - A_1}{A - A_1 + A_2}, \quad \tau_d^2 = \frac{A_2}{A - A_1 + A_2}. \tag{56}$$

If we limit our analysis to signals that vary slowly compared to a half-cycle of the line voltage, then the approximate substitution $e^q - 1 = q$.

We shall replace q by the operator p for a continuous Laplace transform

$$q = \frac{\pi}{\omega_0} p.$$

Then (55) will become

$$Y(p) = K \frac{1 + T_a p + T_b^2 p^2}{1 + T_c p + T_d^2 p^2}, \tag{57}$$

where

$$T_a = \frac{\pi}{\omega_0} \tau_a, \qquad T_b^2 = \left(\frac{\pi}{\omega_0}\right)^2 \tau_b^2, \qquad T_c = \frac{\pi}{\omega_0} \tau_c, \qquad T_d^2 = \left(\frac{\pi}{\omega_0}\right)^2 \tau_d^2. \tag{58}$$

Substituting (56) and (52) into (58) and taking the equations (43) and (44) for the stationary mode into account, we shall find the expressions for the gain and the time constants in terms of the parameters of the basis mode:

$$K = \frac{1}{r'} K'_{id}, \tag{59}$$

where $K'_{id} = \dfrac{\Delta I'_L}{\Delta I'_C}$ is the normalized dynamic current gain.

$$K'_{id} = \frac{\pi}{\pi - \beta} \cdot \frac{1}{1 + \dfrac{1 + \cos\beta}{(\pi - \beta)\sin\beta}},$$
$$T_a = \frac{1}{\omega_0}\left(\pi + \frac{\beta}{2}\right), \qquad T_b^2 = \frac{\pi\beta}{2\omega_0^2}, \tag{60}$$

$$T_{c}=\frac{T_{L}\dfrac{1+\cos\beta}{(\pi-\beta)\sin\beta}+\dfrac{1}{\omega_{0}}\left(\pi+\dfrac{3}{2}\,\dfrac{1+\cos\beta}{\sin\beta}\right)+\dfrac{\pi^{2}}{2\omega_{0}(\pi-\beta)}\,\dfrac{R'}{r'}}{1+\dfrac{1+\cos\beta}{(\pi-\beta)\sin\beta}},$$

$$T_{d}^{2}=\left(\frac{\pi}{\omega_{0}}\right)^{2}\frac{(\omega_{0}T_{L}+\pi-\beta)\left[\dfrac{R'}{2(\pi-\beta)r'}+\dfrac{1+\cos\beta}{2\pi^{2}\sin\beta}\right]}{1+\dfrac{1+\cos\beta}{(\pi-\beta)\sin\beta}}. \qquad (61)$$

We shall use the substitution

$$\frac{I_{L}'}{I_{c.}}=K'_{ist},$$

where K'_{ist} is the normalized static current gain.

In accordance with (46),

$$K'_{ist}=\frac{\pi}{\pi-\beta}. \qquad (62)$$

The quantities K'_{id} and K'_{ist} can be determined from the input-output characteristic of the magnetic amplifier, which is plotted in relative units.

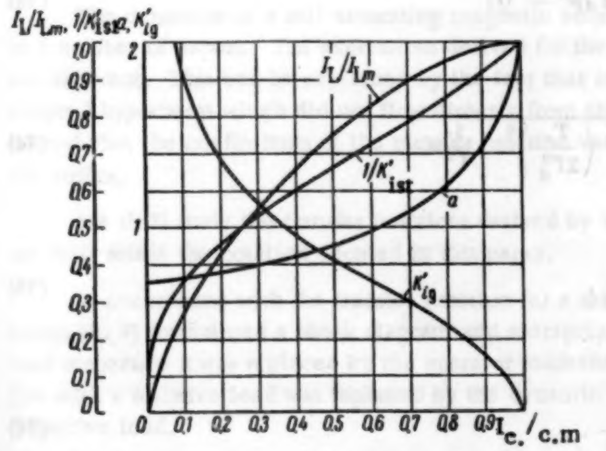

Fig. 5. Graphs of $\underline{a}$, K'_{id}, $1/K'_{ist}$ and I_L/I_{Lm} for an ideal magnetic amplifier.

We shall express the functions of the angle β in (61) in terms of K'_{id} and K'_{ist}. The formulas for the time constants will then become

$$T_{a}=\frac{1}{2f_{0}}\,\frac{3K'_{ist}-1}{2K'_{ist}}, \qquad (63)$$

$$T_{b}^{2}=\frac{1}{8f_{0}^{2}}\,\frac{K'_{ist}-1}{K'_{ist}}, \qquad (64)$$

$$T_{c}=T_{L}a+K'_{id}T_{c.}+\frac{1}{2f_{0}}\left[\frac{K'_{id}}{K'_{ist}}+\frac{3}{2}\,\frac{a}{K'_{ist}}\right], \qquad (65)$$

$$T_{d}^{2}=\left(T_{L}+\frac{1}{2f_{0}K'_{ist}}\right)\left(K'_{id}T_{c.}+\frac{a}{4f_{0}K'_{ist}}\right). \qquad (66)$$

Here

$$a=\frac{K'_{ist}-K'_{id}}{K'_{ist}}, \qquad (67)$$

$$T_{c.}=\frac{1}{4f_{0}}\,\frac{R'}{r'}, \qquad (68)$$

where $T_{c.}$ is the time constant for the control circuit in a magnetic amplifier with a resistive load. The graphs $\underline{a}$, K'_{id} and $(1/K'_{ist})$ computed for an ideal magnetic amplifier from formulas (45), (46), (60), (62) and (67) are shown in Fig. 5.

For sufficiently large T_L and (R'/r'), formulas (65) and (66) are simplified:

$$T_c = T_L a + K'_{id} T_{c.}, \tag{69}$$

$$T_d^2 = T_L T_{c.} K'_{id}. \tag{70}$$

The Transfer Function for the Magnetic Amplifier

The transfer function for the magnetic amplifier expresses its response to a step variation of the input signal for zero initial conditions.

We shall limit our analysis to the case where a) the duration of the transient response appreciably exceeds the time required for a half-cycle of supply voltage and b) the transfer function is expressed in terms of formula (57).

The transform for an input step signal is written as

$$\Delta U'_{c.}(p) = \frac{\Delta U'_{c.}}{p}. \tag{71}$$

The transform of the output signal for zero initial conditions is determined by the product of the transfer function (57) and the transform of the input signal (71).

$$\Delta I'_L(p) = K \Delta U'_{c.} \frac{1 + T_a p + T_b^2 p^2}{(1 + T_c p + T_d^2 p^2)\, p}. \tag{72}$$

The roots of the characteristic equation for the magnetic amplifier are

$$1 + T_c p + T_d^2 p^2 = 0; \tag{73}$$

they can be real and complex:

$$p_{1,2} = -\frac{T_c}{2T_d^2} \pm \sqrt{\left(\frac{T_c}{2T_d^2}\right)^2 - \frac{1}{T_d^2}}. \tag{74}$$

The real roots occur when

$$\alpha > \frac{1}{T_d}, \tag{75}$$

where

$$\alpha = \frac{T_c}{2T_d^2}. \tag{76}$$

We use the substitution

$$\beta = \sqrt{\alpha^2 - \frac{1}{T_d^2}}. \tag{77}$$

We shall find the transfer function for the case of real roots of the characteristic equation by applying the inverse Laplace transform to (72) [7]:

$$\frac{\Delta I'_L(t)}{K \Delta U'_{c.}} = 1 - \frac{(\alpha - \beta) T_b^2 - T_a + \frac{1}{\alpha - \beta}}{2\beta T_d^2} e^{-(\alpha-\beta)t} + \frac{(\alpha + \beta) T_b^2 - T_a + \frac{1}{\alpha + \beta}}{2\beta T_d^2} e^{-(\alpha+\beta)t}. \tag{78}$$

For zero initial conditions a monotonic transient response corresponds to real roots.

When the inequality

$$\alpha < \frac{1}{T_d} \tag{79}$$

is satisfied, the roots of the characteristic equation will be complex.

We shall use the substitution

$$\omega = \sqrt{\frac{1}{T_d^2} - \alpha^2}. \tag{80}$$

Applying the inverse Laplace transform to (72), we shall find the transfer function of the magnetic amplifier for the case of complex roots [7]:

$$\frac{\Delta I_L'(t)}{K\Delta U_{c.}'} = 1 + \frac{1}{\omega T_d}\sqrt{[(\alpha^2-\omega^2)T_b^2 - T_a\alpha + 1]^2 + [T_a - 2\alpha T_b^2]^2\omega^2} \times e^{-\alpha t}\sin(\omega t + \psi), \tag{81}$$

where

$$\psi = \operatorname{arc\,tan}\frac{\omega(T_a - 2\alpha T_b^2)}{(\alpha^2-\omega^2)T_b^2 - T_a\alpha + 1} - \operatorname{arc\,tan}\frac{\omega}{-\alpha}.$$

The complex roots correspond to an oscillatory transient response. H. F. Storm [2, 3] was the first to turn his attention to the possibility that an oscillatory transient response exists in such a magnetic amplifier.

Comparison with Papers by Other Authors

The dynamics of a self-saturating magnetic amplifier with a resistive-inductive dc load has been discussed in a number of papers. The expressions derived for the transfer function of the magnetic amplifier by various authors are different. This can be explained by the fact that in deriving the formula for the transfer function the authors adopted hypotheses which did not flow directly from the fundamental equations for the magnetic amplifier. It is natural that the coefficients of the transfer function varied in accordance with the particular hypothesis adopted by the author.

We shall study the transfer functions derived by various authors as they apply to an ideal magnetic amplifier; we shall retain the notation adopted in this paper.

In accordance with the transfer function for a self-saturating magnetic amplifier with a resistive load, H. F. Storm [2, 3] formulated a block diagram and extrapolated it for the case of a resistive-inductive load when a) the load resistance R was replaced by the operator resistance R + L_p, and b) the current gain $K_i' = 1$ of a magnetic amplifier with a resistive load was replaced by the dynamic current gain K_{id}' of a magnetic amplifier with a resistive-inductive load.

The transfer function of a magnetic amplifier with a resistive-inductive load is given by the system of equations

$$Y(p) = \frac{\Delta I_L'(p)}{\Delta U_{c.}'(p)} = \frac{K_{id}'}{r'}\,\frac{1}{1 + T_c p + T_d^2 p^2}, \tag{82}$$

where

$$T_c = K_{id}' T_{c.}, \tag{83}$$

$$T_d^2 = K_{id}' T_{c.} T_L, \tag{84}$$

with $T_{c.}$ determined by formula (68).

Formulas (82)-(84) can also be obtained by applying the proof proposed by M. A. Rozenblat [4]. The formulas are derived on the assumption that the relationship between the load and control currents remains constant in both the stationary and transient modes. In the book by M. A. Rozenblat, however, an expression for the transfer function which is somewhat different than H. F. Storm's is cited:

$$Y(p) = \frac{I'_L(p)}{U'_c(p)} = \frac{K'_{ist}}{r'} \frac{1}{1 + T_c p + T_d^2 p^2}, \tag{85}$$

where

$$T_c = K'_{ist} T_{c.}, \quad T_d^2 = K'_{ist} T_{c.} T_L. \tag{86}$$

The formulas derived by H. F. Storm are obtained by linearizing the original equations, and the transfer function is derived for the ratio between the increments in the load current and the control voltage. M. A. Rozenblat determines the transfer function for the ratio between the control voltage and the load current; this is permissible only in the case where the coefficients of the differential equation are constant. Moreover, the load current in the magnetic amplifier under study is related to the control current via a nonlinear function.

We shall compare the expressions for the transfer functions (57) and (82). In the transfer function obtained by H. F. Storm there is no polynomial operator in the numerator and there is a substantial difference between the time constant T_c [cf. (83) and (69)] and the value obtained using the formula in this paper. The difference in T_c will have an effect that increases with the amount by which aT_L exceeds $K'_{id}T_{c}$.

In the limiting case where $aT_L \gg K'_{id}T_c$ it follows from (74), (69), (70), (83) and (84) that the roots of the characteristic equation will be equal to

$$p_{1,2} = -\frac{a}{2K'_{id}T_{c.}} \pm \frac{a}{2K'_{id}T_{c.}} \tag{87}$$

(for the expression obtained for the transfer function in this paper) and

$$p_{1,2} = -\frac{1}{2T_L} \pm j\sqrt{\frac{1}{K'_{id}T_{c.}T_L}} \tag{88}$$

(for the transfer function obtained by H. F. Storm).

For a step input signal the roots (87) correspond to an aperiodic transient response, and the roots (88) correspond to an oscillatory response.

L. V. Safris [5] proposed treating the magnetic amplifier as a generator with a certain equivalent emf E and internal resistance R_i, assuming that R_i remains constant and E is related to the control current by the same function in both stationary and transient modes. The internal resistance of the equivalent generator is determined by the partial derivative

$$R_i = \left(\frac{\partial E_0}{\partial I_0}\right)_{I_{c.}=\mathrm{const}}, \tag{89}$$

where E_0 is the average emf in the winding over a half-cycle.

[The transfer function cited in the paper by L. V. Safris is expressed by formula (82).] For the time constants, we cite the formulas

$$T_c = \tau_c + \tau_\nu + \tau_e, \quad T_d^2 = \tau_c\tau_e + \tau_\nu\tau_L, \tag{90}$$

where τ_c, τ_ν, τ_e and τ_L are expressed in terms of the parameters and characteristics of the magnetic amplifier. For an ideal magnetic amplifier, $R_i = \infty$, $\tau_c = 0$, $\tau_\nu = K'_{id}T_c$, $\tau_e = 0$, $\tau_L = T_L$ and

$$T_c = K'_{id}T_{c.}, \quad T_d^2 = K'_{id}T_{c.}T_L. \tag{91}$$

From a comparison of (91) with (83) and (84) it follows that for an ideal magnetic amplifier the transfer functions proposed by L. V. Safris and H. F. Storm are identical.

In contrast to L. V. Safris, Hu Chia-Yao and V. A. Shubenko [6] proposed the computation of the internal resistance of the equivalent generator R_i on the basis of the formula

$$\frac{R}{R_i+R}=\frac{I_L}{mI_{Lm}}=\frac{I'_L}{mI'_{Lm}}, \tag{92}$$

rather than on the basis of (89); here I'_{Lm} is the maximum load current expressed by formula (31) for an ideal magnetic amplifier, and m = 1.05 – 1.15 is an empirical coefficient.

The transfer function of the magnetic amplifier is expressed by formula (82) in accordance with [6], and the time constants are equal to

$$T_c=\tau'_L+\tau_{0(st)}+\tau_{0(\gamma)}, \quad T_d^2=\tau'_L\tau_{0(st)}+\tau_L\tau_0(\gamma). \tag{93}$$

where τ_L, τ'_L, $\tau_{0(st)}$ and $\tau_{0(\gamma)}$ are expressed in terms of the parameters and characteristics of the magnetic amplifier. For the ideal magnetic amplifier,

$$\begin{gathered}\tau_L=T_L,\ \tau'_L=T_L\frac{I'_L}{mI'_{Lm}}, \quad \tau_{0(st)}=0,\ \tau_{0(\gamma)}=K'_{id}T_C. \\ T_c=\frac{I'_L}{mI'_{Lm}}T_L+K'_{id}T_C, \quad T_d^2=K'_{id}T_C T_L.\end{gathered} \tag{94}$$

The time constant T_c differs from the analogous time constant in Storm's formula (83) by a component proportional to the load time constant T_L.

The coefficient I'_L/I'_{Lm} depends on the relative control current and is determined by the input-output characteristic plotted in relative units.

In expression (70) derived above for T_c the load time constant appears with the coefficient $a = K'_{ist} - K'_{id}/K'_{ist}$. The dependence of a and I'_L/I'_{Lm} on the relative control current is expressed using different curves (cf. Fig. 5).

A Comparison of Computed and Experimental Values of the Transient Function

The experiments were performed on a magnetic amplifier with toroidal cores that had the following specifications: The core material was "65NP;" the geometric cross section of one core was 0.4 cm²; the product of the saturation flux density and the space coefficient of the core was $B_sK_{st} = 10.8 \cdot 10^3$ gauss; the rectifiers in the bridge were of the type "D7B"; the number of the turns on the windings was $w_0 = 1000$, $w_C = 2000$; the resistance of the series windings for the two cores was $r_{w_0} = 10.2$ ohm, $r_{w_C} = 101$ ohm.

Experiment No. 1 was performed for parameter values that, according to (79), (76), (65) and (66), assured a transient function that was oscillatory. In experiment No. 1 we assumed the following mode: line frequency $f_0 = 50$ cps, supply voltage U = 18 v, load resistance R = 250 ohm, load time constant $T_L = 51 \cdot 10^{-3}$ sec, control circuit resistance r = 141 ohm.

Fig. 6. Oscillogram of experiment No. 1.

The oscillogram shown in Fig. 6 recorded the load current i_L for a step increase in the control voltage. The step shown in the lower oscillogram trace fixed the instant at which commutation of U_C occurred. The values of the load and control currents were equal to: $I_C = 4.6$ ma, $I_L = 20$ ma (before commutation of U_C) and $I_C = 6.4$ ma, $I_L = 25$ ma (in the steady-state mode after commutation).

The experimentally observed maximum value of the load current was equal to $I_{Lm} = 60$ ma. The maximum value of the control current was $I_{C.m} = I_{Lm}w_0/w_C = 30$ ma.

The initial mode corresponds to a relative control current $I_C/I_{C.m} = 0.154$.

We shall find the computed value of the transient function from the formulas derived in this paper.

For $I_{c.}/I_{c.m} = 0.154$ we find $a = 0.37$, $K'_{id} = 1.51$ and $1/K'_{ist} = 0.42$ from the graphs in Fig. 5. From formulas (63), (64), (68)-(70), (76) and (80) we find: $T_a = 12.9$ millisec, $T_b^2 = 29 \cdot 10^{-6}$ sec², $T_{c.} = 37$ millisec, $T_c = 75$ millisec, $T_d^2 = 2.85 \cdot 10^{-3}$ sec², $\alpha = 13.1$ sec⁻¹, $\omega = 13.3$ sec⁻¹.

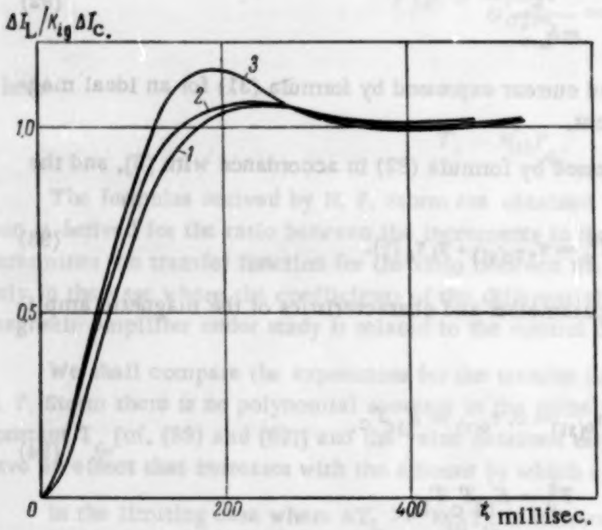

Fig. 7. Transient functions for the mode corresponding to experiment No. 1. 1) Experimental; 2) in accordance with the formulas in this paper; 3) in accordance with the formulas derived by H. F. Storm.

In the investigated mode inequality (79) applies and the transient function is oscillatory. The transient function computed from formula (81) is shown in Fig. 7. The same figure shows the experimental transient function plotted in accordance with the oscillogram in Fig. 6. The computed function is in sufficiently good agreement with the experimental function. A certain divergence can be explained by the effects of the magnetizing current in the operating winding and the value of the ratio $R'/r' = 7.4$ assumed in the investigated mode.

In experiment No. 1 the relative control current corresponds to the intersection points of the curves for a and I_L/I_{Lm} (Fig. 6). Therefore, the values of T_c, computed from formulas (94) and (65), will be almost equal. Since, in addition $T_a < T_c$ and $T_b^2 \ll T_d^2$, it follows that the transient function computed from the Hu Chai-Yao – Shubenko formulas will differ only slightly from the function computed from the formulas in this paper for the investigated mode.

Computation from Storm's formulas (83) and (84) for the mode in experiment No. 1 yields: $T_c = 55.9 \cdot 10^{-3}$ sec and $T_d^2 = 2.85 \cdot 10^{-3}$ sec². For these values of the time constants we have $\alpha = 9.8$ sec⁻¹ and $\omega = 16$ sec⁻¹, in accordance with (76) and (80). The transient function computed from the Storm formulas is shown in Fig. 7. It is also oscillatory in nature but has a somewhat greater over-regulation than the experimental function and the function computed from the formulas in this paper.

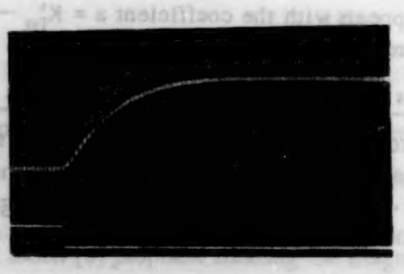

Fig. 8. Oscillogram of experiment No. 2.

Experiment No. 2 corresponded to the following magnetic amplifier mode: $f_0 = 440$ cps, $U = 140$ v, $R = 4000$ ohm, $T_L = 0.1$ sec, $r = 1090$ ohm.

The values of the currents in the initial mode were: $I_{c.} = 10.1$ ma, $I_L = 25$ ma. In the final mode the values were $I_{c.} = 12.5$ ma and $I_L = 28.4$ ma.

The oscillogram of i_L for a step variation of $U_{c.}$ is shown in Fig. 8. In order to set a time scale the signal was modulated at a frequency of 200 cps. From the oscillogram it is evident that the transient function in experiment No. 2 is aperiodic in nature.

We shall find the computed value of the transient function from the formulas in this paper. In the mode corresponding to experiment No. 2, $I_{Lm} = 31.5$ ma, $I_{c.m} = 15.8$ ma, $I_{c.}/I_{c.m} = 0.64$. From the graphs in Fig. 5 we find: $a = 0.505$, $K'_{id} = 0.66$, $1/K'_{ist} = 0.75$.

From formulas (63)-(66), (68), (76), and (77) we find $T_a = 1.28$ millisec, $T_b^2 = 0.16 \cdot 10^{-6}$ sec², $T_{c.} = 8.35$ millisec, $T_c = 57.2$ millisec, $T_d^2 = 0.58 \cdot 10^{-3}$ sec², $\alpha = 49.5$ sec⁻¹, $\beta = 26.8$ sec⁻¹.

In the investigated mode inequality (75) applies and the transient function is aperiodic in nature. The transient function computed from formula (78) is shown in Fig. 9. The same figure also shows the experimental functions. The computed function shows very good agreement with the experimental function.

We shall find the transient function from the Hu Chia-Yao – Shubenko formulas.

For $I_c/I_{c.m} = 0.64$ we have $I_L/I_{Lm} = 0.85$, in accordance with Fig. 5. According to (94) we have $T_c = 90.5$ millisec and $T_d^2 = 0.551 \cdot 10^{-3}$ sec^2 for $m = 1$. In accordance with (76) and (77), $\alpha = 82$ sec^{-1} and $\beta = 70$ sec^{-1}. Under these conditions inequality (75) is satisfied and the transient function is aperiodic (Fig. 9). Since in the chosen mode $I_L/I_{Lm} > a$, the time constant T_c computed from the Hu Chai-Yao – Shubenko formula has a value that is too high. The transient process expressed by the computed function proceeds more slowly than that observed experimentally. By the introduction of the empirical coefficient $m = 1.05 - 1.15$ into the formula for T_c proposed by the authors it is evident that we cannot bring T_c to the level defined by the more rigorous theory. Note that in the region of small relative control currents ($I_c/I_{c.m} < 0.15$) the correction coefficient must be less than unity.

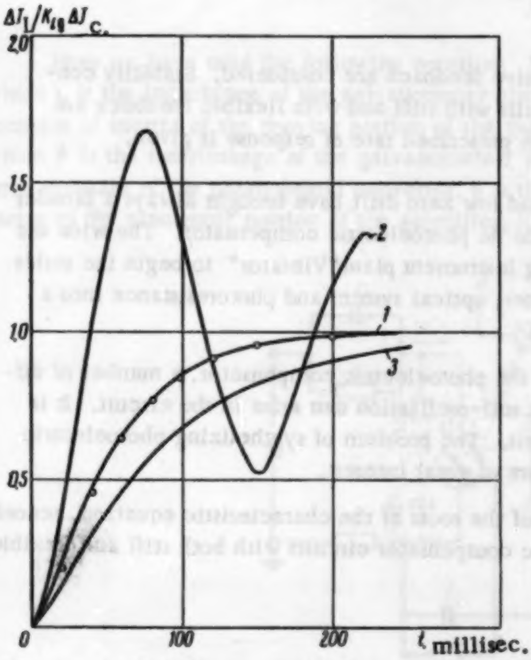

Fig. 9. Transient functions for the mode corresponding to experiment No. 2. 1) Experimental function; 2) function derived in accordance with H. F. Storm's formulas; 3) function derived according to the Hu Chia-Yao – Shubenko formulas; the open circles denote the points computed from the formulas in this paper.

We shall find the transient function from the Storm formulas. In accordance with (83) and (84) we have $T_c = 5.5 \cdot 10^{-3}$ sec, $T_d^2 = 0.55 \cdot 10^{-3}$ sec^2. Under these conditions inequality (79) is satisfied and the computed transient function is oscillatory in nature. In accordance with (76) and (80), $\alpha = 5$ sec^{-1} and $\omega = 42.4$ sec^{-1}. The transient function computed according to the Storm method for the mode in experiment No. 2 is shown in Fig. 9. It diverges sharply from the experimental function which is aperiodic.

CONCLUSIONS

1. For an ideal self-saturating magnetic amplifier with a resistive-inductive load the authors derive a system of equations in finite differences that determines the relationship between the step functions for the control voltage and load current.

2. For the case where $\omega_0 T_L \gg 1$ and $r' \ll R'$, a simple expression is derived for the transfer function of the magnetic amplifier in terms of the parameters of the stationary mode.

3. Expressions are derived for the transient function of the magnetic amplifier. The relationships are found between the parameters corresponding to an aperiodic or oscillatory transient function.

4. Experiment has shown that there is good agreement between the computed and experimental values of the transient function in different modes.

5. Formulas are investigated for the transfer functions derived by a number of authors. Parameter regions are defined within which the formulas will produce an appreciable divergence from experimental results.

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ON THE DYNAMICS OF PHOTOELECTRIC COMPENSATORS

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The dynamics of photoelectric dc compensators with negative feedback are considered. Stability conditions and a comparison of stabilization methods for circuits with stiff and with flexible feedback are given. A possible approach to the synthesis of circuits with prescribed rate of response is given.

The presence of high-amplification (up to 10^8 in voltage) and low zero drift have brought always a broader recognition in measurement techniques and automation [1-4] to the dc photoelectric compensator. The wide use of these instruments has caused the Leningrad electrical measuring instrument plant "Vibrator" to begin the series production of photoelectric compensators combining a galvanometer, optical system and photoresistance into a single design unit, in 1957.

However, in the construction of specific circuits containing the photoelectric compensator, a number of difficulties are frequently encountered. Due to presence of feedback, self-oscillation can arise in the circuit. It is difficult to achieve a satisfactory rate of response in a stable circuit. The problem of synthesizing photoelectric compensator circuits with prescribed dynamic properties is therefore of great interest.

The method of calculation here is based on the assignment of the roots of the characteristic equation, according to T. N. Sokolov [5], and permits the synthesis of photoelectric compensator circuits with both stiff and flexible negative feedbacks.

Photoelectric Compensator with Feedback

The schematic diagram of a photoelectric compensator is given in Fig. 1. The sensitive element is a moving coil galvanometer G using tension suspension. The shift of the galvanometer light spot is transformed by a photo-optical converter using photoelements P_1 and P_2 and a vacuum tube amplifier, to a variation of anode current I in the vacuum tube T. Resistances r_1–r_6 and r_9, together with capacitors C_1–C_6 form a stabilizing circuit, used to obtain the required circuit characteristics.

Feedback can be realized both through capacitor C_7 and resistances r_7 and r_8 (flexible feedback), and directly through resistance r_7 (stiff feedback).*

A block diagram of the compensator is given in Fig. 2. The stabilizing circuit, which, as can be seen from Fig. 1, can be connected in various ways, is here assigned to the compensator input by convention. The transfer functions of the individual circuits are given below:

Electrical input circuit of the compensator

$$\frac{i(p)}{e(p)} = \frac{1}{\Sigma r(\tau p + 1)},$$

Galvanometer

$$\frac{\alpha(p)}{i(p)} = \frac{1}{Jp^2 + Pp + W_M},$$

Photooptical converter

$$\frac{U_1(p)}{\alpha(p)} = \frac{\psi}{1 + T_y p},$$

* Circuits with stiff and with flexible feedback will be considered separately below. Combined feedback, which is rarely employed, will be omitted in the present article.

Stabilizing circuit

$$\frac{U_2(p)}{U_1(p)} = Y_{st}(p),$$

Vacuum tube amplifier

$$\frac{I(p)}{U_2(p)} = S(p),$$

Feedback circuit

$$\frac{e_{fb}(p)}{I(p)} = k_{fb}(p).$$

Here we have used the following notation: Σr is the sum of resistances in the galvanometer circuit; $\tau = L/\Sigma r$, where L is the inductance of the galvanometer circuit; α is the angle of rotation of the galvanometer coil; J is the moment of inertia of the moving portion of the galvanometer; $P = \psi^2/\Sigma r$ is the damping factor of the galvanometer, where ψ is the interlinkage of the galvanometer; W_M is the constant tensile moment of the galvanometer; T_y is the time constant of the photooptical converter, k is the conversion factor of the photooptical converter; S(p) is the gain factor of the electronic portion of the amplifier, $k_{fb}(p)$ is the transfer factor of the feedback circuit.

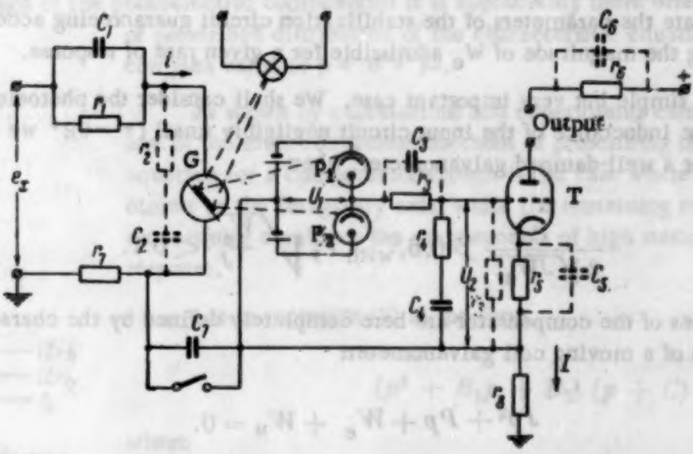

Fig. 1. Schematic diagram of photoelectric compensator with stabilizing networks.

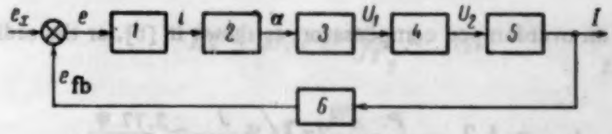

Fig. 2. Block diagram of photoelectric compensator. 1) Amplifier input electrical network; 2) galvanometer; 3) photooptical converter; 4) stabilization network; 5) electronic amplifier; 6) negative feedback circuit.

The transfer function of the closed loop is

$$\frac{I(p)}{e_x(p)} = \frac{k_p Y_{st}(p)}{(1+T_y p)(1+\tau p)(Jp^2+Pp+W_M)+k_p Y_{st}(p)\, k_{fb}(p)}, \quad (1)$$

where $k_p = \frac{\psi k S}{\Sigma r}$. Here we have assumed that S(p) = S, i.e., the electronic amplifier has no lag.

In the case of stiff feedback, as is evident from Fig. 1:

$$k_{fb}(p) = \frac{r_7 r_8}{r_7 + r_8} = r_k. \quad (2)$$

Putting $k_p r_k = W_e$, we obtain from (1)

$$\frac{J(p)}{e_x(p)} = \frac{k_p Y_{st}(p)}{(1 + T_y p)(1 + \tau p)(Jp^2 + Pp + W_M) + W_e\, Y_{st}(p)}. \quad (3)$$

In each concrete case the compensator dynamics can be calculated from the measured parameters of the galvanometer J, P (ψ), W_M and the time constants T_y and τ, which can be calculated or found experimentally. The value $r_k \approx e_x/I$ is found on the basis of the desired amplifier sensitivity. The quantity W_e characterizes the static error of the compensator with the given W_M. The greater W_e is, the smaller [as can be seen from (3)] is the error. Simultaneously the noise connected with power supply instability, photoelements, etc., is reduced.

However, as W_e is increased (for example, by increasing S) the tendency to oscillate will increase sharply. Therefore in calculating the dynamics of photoelectric compensators with stiff feedback two procedures should be used: For a given W_e calculate the parameters of the stabilization circuit guaranteeing acceptable rate of response, or, on the contrary, defining the magnitude of W_e admissible for a given rate of response.

Let us consider first a simple but very important case. We shall consider the photoelectric converter to be without lag ($T_y = 0$), and the inductance of the input circuit negligibly small ($\tau = 0$);* we also put $Y_{st}(p) = 1$. As shown in [2], this is valid for a well-damped galvanometer, when

$$\frac{P}{2\sqrt{JW_e}} > 0.6, \text{ with } T_y\sqrt{\frac{W_e}{J}} < 0.1.$$

The dynamic properties of the compensator are here completely defined by the characteristic equation, analogous to the ordinary equation of a moving coil galvanometer:

$$Jp^2 + Pp + W_e + W_M = 0. \quad (4)$$

Practically always $W_e \gg W_M$ and the equation takes the form

$$Jp^2 + Pp + W_e = 0. \quad (5)$$

The damping time** of an overdamped compensator, as shown in [6], for the ordinary galvanometer, is in this case defined by the formula

$$t_d \approx 1.2\frac{P}{2\sqrt{JW_e}}\, 2\pi\sqrt{\frac{J}{W_e}} \approx \frac{3.77 \cdot \psi}{k S r_k}. \quad (6)$$

A family of curves $t_d = f(r_k)$ with kS = const for a given galvanometer is given in Fig. 3. The lines Σr mark off the regions of overdamping. The solution of (6) is valid for given Σr in the region above the straight lines. Since r_k is found by calculating the static behavior of the photoelectric compensator, these curves give the possibility of determining the required value of kS for a given t_d.

The above case of overdamped amplifier is possible in reality only when it is possible to accept a relatively low value of W_e, for example, in the design of high-sensitivity measuring instruments. The requirement of increasing the static precision and stability of the photoelectric compensator, for example, in the design of precise voltage and current stabilizers, inevitably leads to increase of W_e and the passage of the amplifier to the under-

* In practice the quantity τ does not exceed 10^{-3}–10^{-4} sec.

** By damping time we understand the time from the instant of application of a step e_x at the input to the instant where the output current will differ by not more than 2% from the steady-state value.

damped condition. In this case the dynamic properties of the photoelectric compensator are defined by a third-order characteristic equation, and the larger of the time constants T_y or τ * must be included in the calculation,

$$(Jp^2 + Pp + W_M)(1 + T_y p) + W_e = 0. \tag{7}$$

Considering that $W_M T_y \ll P$ and $W_M \ll W_e$, we obtain

$$p^3 + \frac{PT_y + J}{JT_y} p^2 + \frac{P}{JT_y} p + \frac{W_e}{JT_y} = 0. \tag{8}$$

It should be emphasized, contrary to the opinion of certain authors [4], that in calculating the photoelectric compensator dynamics it is always possible in practice to neglect the quantity W_M.

The stability condition from Eq. (8) is

$$JT_y W_e < P(PT_y + J). \tag{9}$$

For given numerical values of the gain factors, it is not difficult to find the roots of the characteristic equation for a stable system, using one of the known methods, for example, the method developed by D. A. Bashkirov [7]. However in the design of the photoelectric compensator it is appreciably more effective to use the method of prescribed distribution of the characteristic equation roots in the plane of the complex variable $p = \sigma + j\omega$.

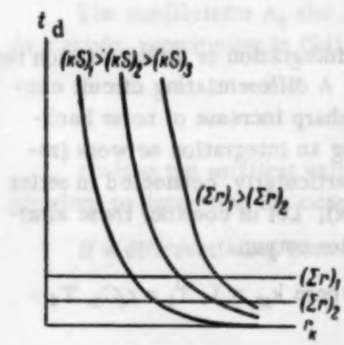

Fig. 3. Determining the damping time of an overdamped photoelectric compensator.

As shown by calculations and experiments carried out by the present author, one of the most convenient methods of prescribing the roots of the characteristic equation for a compensator system is the case where a pair of complex roots is closest to the imaginary axis, while the remaining roots are real (Fig. 4). This successfully combines the requirements of high static precision and good rate of response.

Let us represent (8) in the form

$$(p^2 + B_1 p + B_2)(p + C) = 0, \tag{10}$$

where

$$B_1 + C = \frac{PT_y + J}{JT_y} = A_1, \tag{11}$$

$$B_2 + B_1 C = \frac{P}{JT_y} = A_2, \tag{12}$$

$$B_2 C = \frac{W_e}{JT_y}. \tag{13}$$

Since the transient process in the system is defined by the pair of complex roots, the relationship between the coefficients B_1 and B_2 should be prescribed:

$$B_2 = k_n B_1^2 = (1 \text{ to } 4)\, B_1^2. \tag{14}$$

This corresponds to a damped transient process in the system during 1-2.5 periods of output current oscillation. Putting $B_1 = aA_1$, we obtain $C = (1 - a)\, A_1$. Using (11) and (12), we obtain

$$a^2 (k_n - 1) + a - \frac{A_2}{A_1^2} = 0. \tag{15}$$

* In using vacuum photoelements with high internal resistance ρ this constant can also be the input time constant of the vacuum tube $\tau_{in} = \rho C_{in}$, where C_{in} is the input capacitance of the vacuum tube ($\tau_{in} \approx 10^{-1}$–$10^{-2}$ sec).

From this we find the smallest value of $\underline{a}$ (the coefficients A_1 and A_2 are given by the galvanometer and amplifier design) and further, B_2, C and W_e.

To estimate the damping time, the approximate formula

$$t_d \approx k_t \frac{2\pi}{\sqrt{B_2}} \tag{16}$$

may be used, where k_t depends on k_n' (with $k_n = 1$, $k_t = 1$). It can be stated that a system designed in this way will have the maximum value of W_e for a given rate of response.

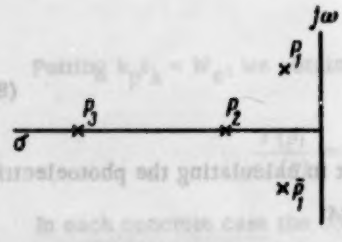

Fig. 4. Distribution of the roots of the characteristic equation of a closed-loop system in the complex plane.

Stabilization Circuit of Photoelectric Compensator with Flexible Feedback

To increase the rate of response and the permissible value of W_e many authors [3, 4, 8, 9] have proposed a number of compensator stabilization circuits, the basic ones of which being given in Fig. 1.*

As is easily verified, the transfer function of the stabilization network in almost all cases has the form

$$Y_{st}(p) = k_{st} \frac{1 + T_1 p}{1 + T_2 p}. \tag{17}$$

Depending on the relationship between T_1 and T_2 the stabilizing circuit is an integration or differentiation network. However the point at which the network is introduced is far from unimportant. A differentiating circuit cannot be used at the input to the vacuum tube amplifier or in the first stage, due to a sharp increase of noise background at the compensator output. The best results are therefore given by connecting an integration network (resistances r_3, r_4, and capacitor C_4) in the grid circuit of the input vacuum tube and particularly, connected in series with the galvanometer resistance r_1, shunted by capacitor C_1 (differentiating network). Let us consider these alternatives in greater detail, since they do not increase the noise level at the compensator output.

When the integration network is used, its transfer function has the form (17), where $k_{st} = 1$, $T_1 = r_4 C_4$, $T_2 = (r_3 + r_4) C_4$.

The use of this network is particularly advantageous at high gain S, to suppress power line hum arising in the grid circuits in the presence of high resistance vacuum photoelements.

Let us write the characteristic equation of the closed loop (with the ratio $W_e T_1 > P$ applying in practice), putting $W_M = 0$:

$$p^4 + \frac{J(T_y + T_2) + PT_y T_2}{JT_y T_2} p^3 + \frac{J + P(T_y + T_2)}{JT_y T_2} p^2 + \frac{W_e T_1}{JT_y T_2} p + \frac{W_e}{JT_y T_2} = 0. \tag{18}$$

The stability condition of the circuit is

$$T_1 \left\{ \frac{[J(T_y + T_2) + PT_y T_2][J + P(T_y + T_2)]}{JT_y T_2} - W_e T_1 \right\} - \frac{[J(T_y + T_2) + PT_y T_2]^2}{JT_y T_2} > 0. \tag{19}$$

Similarly to the above, the roots of the characteristic equation may be found. Strictly speaking, to investigate the dynamics of a compensator with stabilization networks, attention should not be limited to the characteristic equation but the transfer function of the total system should be taken into account, since the factor $T_1 p + 1$ appears in its numerator. However it can be shown that in the ordinary range of frequencies in the compensator (0-10 cps)

* The photoelectric compensator circuits not recommended by the present author are shown in Fig. 1 by broken lines.

this limitation is hardly essential. Therefore, for synthesis purposes we represent (18) in the form

$$(p^2 + B_1 p + B_2)(p + C)(p + D) = 0. \tag{20}$$

For the integrating network usually

$$T_2 > T_y, \tag{21}$$

$$PT_2 > J, \tag{22}$$

$$B_1 + C + D = \frac{J + PT_y}{JT_y} = A_1, \tag{23}$$

$$B_2 + B_1(C + D) + CD = \frac{P}{JT_y} = A_2, \tag{24}$$

$$B_2(C + D) + B_1 CD = \frac{W_e T_1}{JT_y T_2}, \tag{25}$$

$$B_2 CD = \frac{W_e}{JT_y T_2}. \tag{26}$$

The coefficients A_1 and A_2 are given. Putting $B_1 = C = aA_1$, we obtain $D = A_1(1 - 2a)$, $B_2 = k_n B_1^2 = k_n a^2 A_1^2$. As a result, substituting in (24), we obtain

$$k_n a^2 A_1^2 + aA_1^2(1 - a) + aA_1^2(1 - 2a) = A_2. \tag{27}$$

Finding the smallest value of $\underline{a}$ from (27) and substituting B_1, B_2, C and D in (25) and (26), we obtain two expressions to determine the constants T_1 and T_2 with given W_e.

If a differentiating network is used for stabilization in the galvanometer circuit, then

$$k_{st} = \frac{\Sigma r}{\Sigma r + r_1}, \qquad T_1 = r_1 C_1, \qquad T_2 = k_{st} T_1.$$

Then the simplifying assumptions (21), (22) may be found invalid, and therefore for synthesis purposes it is necessary to consider the coefficients of Eq. (18) in general form. In this case to solve the system it is most suitable to use a method of successive approximations. For example, assigning $a = 0, 1$, $k_n = 1$ and $B_1 = C$, we obtain from the first of the two equations [of type (23) and (24)] the numerical values of A_1 and T_2, and thus of all the remaining coefficients. Substituting the expressions obtained in the last equation, a certain difference is found between the left- and right-hand sides. Changing the value of $\underline{a}$, we find the new values of the coefficients forming an equation of type (26) identically. Finally, the constant T_1 is found from the next to the last equation.

Synthesis of Photoelectric Compensators with Flexible Feedback

A photoelectric compensator with flexible feedback is an interesting instrument, analogous to the well-known electronic integrator circuit.

The principle difference consists in the substitution for the input stages of the vacuum tube amplifier of the photooptical converter, which permits integration of vanishingly-small input signals ($\sim 10^{-8}$ v). Particular interest is presented by the application of this type of instrument to measure small magnetic fluxes [10, 11].

In this case formula (2) takes the form (see Fig. 1)

$$k_{fb}(p) = \frac{T_3 p}{T_4 p + 1}, \tag{28}$$

where $T_3 = r_7 r_8 C_7$, $T_4 = (r_7 + r_8) C_7$.

The transfer function of the closed loop of system (1), using a stabilization network with transfer function (17), taking into account (28), has the form

$$\frac{I(p)}{e_x(p)} = \frac{k_p(1+T_1p)(1+T_4p)}{(Jp^2+Pp+W_M)(1+T_yp)(1+T_2p)(1+T_4p)+k_pT_3p(1+T_1p)}. \tag{29}$$

Similarly to the above, the greater of the constants τ or T_y should be substituted in formula (29). Putting $W_M = 0$, which, as can be shown, is equivalent to the simple dropping of integration "creep," we obtain the characteristic equation

$$p\left[p^4 + \frac{JT_y(T_2+T_4)+T_2T_4(J+PT_y)}{JT_yT_2T_4}p^3 + \frac{JT_y+PT_2T_4+(J+PT_y)(T_2+T_4)}{JT_yT_2T_4}p^2 + \frac{k_pT_1T_3+P(T_y+T_2+T_4)+J}{JT_yT_2T_4}p + \frac{k_pT_3+P}{JT_yT_2T_4}\right] = 0. \tag{30}$$

The difference in the system dynamics from those of an ideal integrator is due to the expression in square brackets. In this case the stability condition will be the inequality*

$$T_2[JT_y+T_4(J+PT_y)]\{[J+P(T_y+T_4)]T_1 - [JT_y+T_4(J+PT_y)]\} - k_pJT_yT_1^2T_3T_4 > 0. \tag{31}$$

In the absence of a stabilization network ($T_1 = T_2 = 0$) Eq. (3) takes the form

$$p^3 - \frac{PT_yT_4+J(T_y+T_4)}{JT_yT_4}p^2 + \frac{J+P(T_y+T_4)}{JT_yT_4}p + \frac{k_pT_3}{JT_yT_4} = 0. \tag{32}$$

We shall consider T_3 and T_4 prescribed, since the constant T_3 is given by the desired transfer factor of the integrator while T_4 is rigidly related to T_3. In this case the synthesis by Eq. (32) in no way differs from the above investigation of Eq. (8) formulas (10)-(16)]. The role of W_e is here played by k_p.

In the general case, to increase k_p, it is necessary to employ stabilization. Representing (30) in the form (20) and taking into account the remarks concerning inequality (31), we obtain

$$B_1 + C + D = \frac{JT_y + T_4(J+PT_y)}{JT_yT_4} = A_1, \tag{33}$$

$$B_2 + B_1(C+D) + CD = \frac{J+P(T_y+T_4)}{JT_yT_4} = A_2, \tag{34}$$

$$B_2(C+D) + B_1CD = \frac{k_pT_1T_3}{JT_yT_2T_4}, \tag{35}$$

$$B_2CD = \frac{k_pT_3}{JT_yT_2T_4}. \tag{36}$$

Since coefficients A_1 and A_2 are prescribed, the calculation of the system is analogous to that based on Eq. (18) for the compensator system with rigid feedback [Eq. (20)-(27)].

In conclusion, let us consider an example of photoelectric compensator calculation.

Let it be required to calculate a photoelectric compensator to stabilize a voltage, maintaining the output voltage constant over a long period to a precision of 0.002%, by means of stiff feedback. The galvanometer used for constructing the system has the following parameters: $J = 0.2$ g·cm²; $P = 80$ d·cm·sec; $W_M = 3$ d·cm/rad.

* In composing (31) the following, practically always valid assumptions were made: $k_pT_3 > P$, $T_2 > T_4$, $T_2 > T_y$, $k_pT_1T_3 > P(T_y + T_2 + T_4) + Z$.

The time constant of a cadmium-sulfide photoresistance $T_y = 0.01$ sec. To provide the required output voltage stability the value W_e calculated by the method described in [2] should be not less than 45,000 d·cm/rad.

As is evident from formula (9), without the use of a stabilization network the system will be unstable. Stable operation of the system at maximum speed can be obtained by using a differentiating network in the galvanometer circuit. However here the galvanometer circuit will have an additional resistance and to obtain the required value of W_e a further increase of amplification will be necessary. Therefore, limiting ourselves to the use of an integrating network and putting $k_n = 1$, we obtain from (23), (24) and (27): $A_1 = 500\ \text{sec}^{-1}$; $A_2 = 40{,}000\ \text{sec}^{-2}$; $a = 0.09$; $B_1 = C = 45\ \text{sec}^{-1}$; $B_2 = 2{,}000\ \text{sec}^{-2}$; $D = 410\ \text{sec}^{-1}$.

On the basis of (25) and (26) this gives $T_2 \approx 0.6$ sec, $T_1 \approx 0.05$ sec.

Now the damping time calculated from the approximate formula (16), $t_d \approx 0.14$ sec. The actual damping time of the system constructed on the basis of the above calculation was about 0.12 sec.

CONCLUSIONS

The synthesis of photoelectric compensators with prescribed response rate can be realized using prescribed distribution in the complex plane of the roots of the characteristic equation of the closed-loop system. This method leads most simply to design formulas for the stabilization network parameters.

Those stabilization circuits for the photoelectric compensator can be recommended which use an integrating network at the input to the vacuum amplifier (circuits with stiff and with flexible feedback) or a differentiating network in the galvanometer circuit (circuits with stiff feedback).

The author expresses appreciation to S. G. Rabinovich for the valuable advice.

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APPLICATION OF THE PRINCIPLE OF INVARIANCE TO THE NONLINEAR ACTION RESULTING FROM A DISTURBANCE

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We examine the realization of the principle of invariance in the case of the nonlinear action $Y = -A_0 \sin(Df)$ resulting from the disturbance $f(t)$. We show that to compensate for the distrubance we must provide a nonlinear transducer in a parallel channel, which supplies a trapezoidal signal.

We formulate the conditions which assure the realization of the principle of invariance for the given class of systems.

The theoretical analysis is verified by means of an electrical model.

Following the theory outlined by Academician B. N. Petrova [1] we examine the possibility of realizing the principle of invariance in the case of a nonlinear disturbance $f(t)$ acting upon an object that is subject to regulation.

We will assume that the system whose structure is outlined in Fig. 1 exists. This system contains a parallel channel which has been built for the purpose of compensating for the action $Y = \varphi(f)$ resulting from the disturbance $f(t)$.

The dynamic properties of the elements in the parallel channel are given by the following equations:

for the measuring element

$$P_{Y1}Y_1 = Q_{Y1}f(t),$$

for the nonlinear portion of the transducer

$$Y_2 = \psi(Y_1),$$

for the linear portion of the transducer

$$P_{Y3}Y_3 = Q_{Y3}Y_2,$$

for the regulating element

$$P_{Z1}Z_1 = Q_{Z1}X_2,$$

where P_i and Q_i are polynomials obtained from the differential operator $D = d/dt$.

It follows from these equations that the action Z_1 in the parallel channel, upon the regulated object resulting from the disturbance $f(t)$ is given by (for $X_1 = 0$)

$$Z_1 = \frac{Q_{Z1}Q_{Y3}}{P_{Z1}P_{Y3}}\psi\left(\frac{Q_{Y1}}{P_{Y1}}f\right). \tag{1}$$

Invariance of the regulated coordinate X with respect to the disturbance $f(t)$ will be assured if

$$Z = Y + Z_1 = \varphi(f) + \frac{Q_{Z1}Q_{Y3}}{P_{Z1}P_{Y3}}\psi\left(\frac{Q_{Y1}}{P_{Y1}}f\right) = 0. \tag{2}$$

Equation (2) will hold if the two conditions

$$Q_{Z1}Q_{Y3} = -P_{Z1}P_{Y3} \quad \text{or} \quad \frac{Q_{Z1}Q_{Y3}}{P_{Z1}P_{Y3}} = k \tag{3}$$

and

$$\varphi(f) = -\psi\left(\frac{Q_{Y1}}{P_{Y1}}f\right) \quad \text{or} \quad \varphi(f) = -k\psi\left(\frac{Q_{Y1}}{P_{Y1}}f\right) \tag{4}$$

are fulfilled simultaneously.

Condition (3) states that the transfer function of the linear portion of the transducer element must be an inverse transfer function of the regulatory unit (or element). In order to fulfill condition (4) the transducer must create a nonlinearity of the same form as that of the nonlinear disturbance acting on the regulated object.

One of the nonlinear disturbances acting upon the regulated object that is often met with in practice is of the form

$$\varphi(f) = -A\,\mathrm{sign}\,(Df). \tag{5}$$

Equation (5) determines, for example, the resultant action due to dry friction in the support (with a degree of accuracy that is adequate for engineering calculations).

In that case,in order to satisy Eq. (4) we must have

$$\frac{Q_{Y1}}{P_{Y1}} = cD, \tag{6}$$

where c is a constant coefficient.

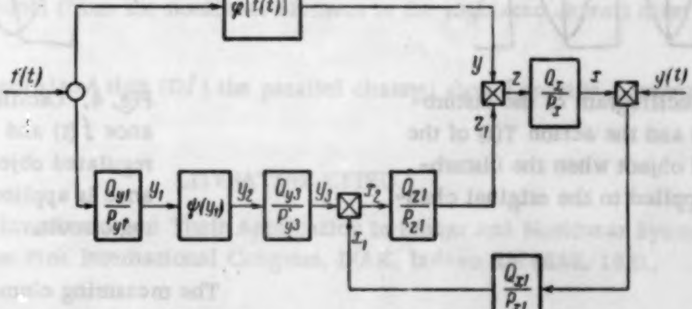

Fig. 1. Block diagram of the system.

We can realize condition (6) in many ways. The basic method involves the use of a measuring device whose signal is proportional to the derivative of the disturbance f(t). However,actual differentiating elements (differentiators) usually possess inertia, therefore we must have an active forcing circuit at the output of the measuring device.

In the majority of cases the regulating element of the system is an inertial element and therefore an active forcing circuit at the transducer output is required in order to fulfill condition (3).

If we use a circuit containing electron tubes and semiconductors it is not difficult to obtain the nonlinearity defined by Eqs. (5). However if we do not satisfy Eq. (3) precisely or if the parameters of the elements involved change during operation we may get a phase shift in the phase of Y or Z_1 that will differ from 180°. As a result of this we get an impulse, due to the total action of the disturbance, of amplitude 2A.

It is therefore more desirable to design a transducer which supplies a nonlinear trapezoidal signal. In the latter case even though we cannot avoid the action of the total disturbing forces upon the object their amplitude will not

significantly exceed A. Also, for every change in the sign of Y_1 we will have two impulses of opposite sign, one after the other.

The theory outlined above was verified by means of the electronic-analog (electronic model) device ÉMU-8.

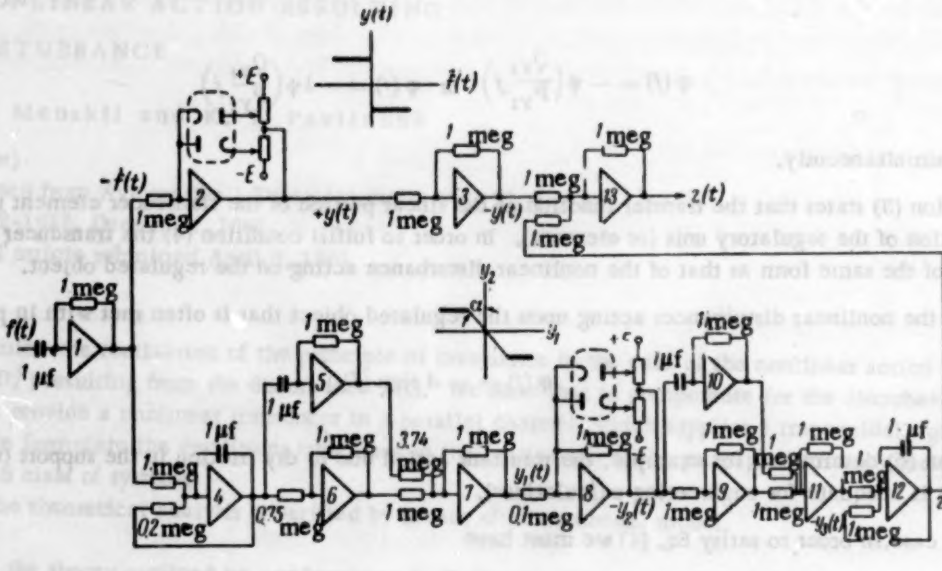

Fig. 2. Block diagram of the electronic model of the problem as set up on the ÉMU-8.

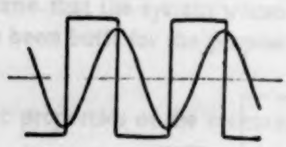

Fig. 3. Oscillogram of the disturbance f(t) and the action Y(t) of the regulated object when the disturbance is applied to the original channel.

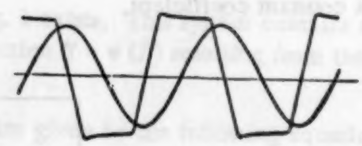

Fig. 4. Oscillogram of the disturbance f(t) and the action Z_1(t) of the regulated object when the disturbance is applied to the compensating circuit.

Fig. 5. Oscillogram of the disturbance f(t) and the total action Z(t) of the regulated object for a trapezoidal compensating signal.

The measuring element was modeled by a real differentiator having the transfer function

$$W'_1 = \frac{k_1 D}{T_1 D + 1},$$

where k_1 = 7.5 and T_1 = 0.02 sec.

In order to satisfy condition (6) we place a focusing circuit at the input of the measuring element. The transfer function of the measuring element is

$$W''_1 = k_2 (T_1 D + 1),$$

where k_2 = 0.133.

Therefore

$$\frac{Q_{Y1}}{P_{Y1}} = W'_1 W''_1 = k_1 k_2 D.$$

The signal from the nonlinear portion of the transducer is selected by the trapezoidal form. The linear part of the transducer output is represented by a forcing link of the first order,

$$\frac{Q_{Y_3}}{P_{Y_3}} = k_3(T_2D + 1),$$

where $k_3 = 0.895 \cdot 10^{-4}$ and $T_2 = 0.05$ sec.

The value of the transfer function Q_{Y_3}/P_{Y_3} is determined from (3).

We took as the transfer function of the regulating link

$$\frac{Q_{Z1}}{P_{Z1}} = \frac{k_4}{T_3D + 1},$$

where $k_4 = 1.12 \cdot 10^4$.

The schematic diagram of the electronic analog (model) for the problem is shown in Fig. 2.

As the external disturbance $f(t)$ we took a harmonically changing signal. The effect upon the system of the action of the disturbance along the original channel is shown in Fig. 3; the signal created by the compensating channel is shown in Fig. 4.

The oscillogram (Fig. 5) when the disturbance is applied to both channels shows that condition (4) is fulfilled assuring complete compensation over the greater portion of the period for the disturbance.

Experimental verification showed that under the action of random disturbances (i.e., of disturbances with a random character) and the associated conditions of exploitation of the system, the presence of a compensating channel cuts the mean dynamic error in half even when condition (3) is not precisely satisfied.

CONCLUSIONS

In order to compensate for nonlinear reactions to disturbances affecting the regulated object it is necessary to introduce a transducer into the parallel channel which possesses the same type of nonlinearity. In addition the transfer function for the parallel channel (from the nonlinear element to the regulated object) must be equal to k (where k is a constant).

If the disturbance $f(t)$ is equal to A sign (Df) the parallel channel should provide a trapezoidal signal.

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ON THE BASIS OF AN APPROXIMATE METHOD OF INVESTIGATING TRANSIENT PROCESSES IN POST-ACTION AUTOMATIC CONTROL SYSTEMS

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In this paper we examine the questions associated with the bases underlying an approximate method of investigating transient processes in post-action automatic control systems (ACSF). In this method we take into account only a finite number of non-asymptotic roots λ_j ($j = 1,2,\ldots,m$) of the characteristic quasipolynomial [1]. The latter has, generally speaking, an infinite number of roots. In this article we evaluate the influence upon the transient process of the neglected roots. We will also study the questions involving the phenomenon of "heredity" in (ACSF).

1. Qualitative Theorems Regarding the Evaluation of the Increase in the Solution of the Differential Equation of a Retarded Variable and its Application to the Approximate Study of Transients in ACSF

The theorems regarding the evaluation of the increase in the solution of the differential equation of a retarded variable of the form

$$y^{(n)}(t) + \sum_{\nu=1}^{n} [a_\nu y^{(n-\nu)}(t) + b_\nu y^{(n-\nu)}(t - \tau_\nu)] = f(t), \tag{1}$$

where a_ν and b_ν are constant coefficients and τ_ν satisfies the inequality $0 < \tau_\nu \leq \delta < \infty$ stated in E. Pinny's book [2] give us extremely useful information regarding the behavior of the solution y(t) as $t \to \infty$. We will state these theorems without proof.

Theorem 1. (E. Pinny). Let us consider an equation of a retarded variable of the form (1) for which the condition $y(t) = \varphi(t)$ is satisfied over the initial interval $-\delta \leq t < 0$ where $\varphi(t)$ is bounded, satisfies the Dirichlet conditions and is a (n-1)st differentiable function. In addition we are given the value $y^{(\nu)}(t_0)$ ($\nu = 0,1,\ldots,n-1$) to enable us to fix the point t_0. Further, $f(t)$ and its derivatives and $y^{(\nu)}(t)$ increase less rapidly than

$$\begin{aligned} |f(t)| &< M\left(1 + \frac{t}{\delta}\right)^p e^{kt} && \text{for } t \geqslant 0, \\ |f'(t)| &< M\delta^{-1}\left(1 + \frac{t}{\delta}\right)^p e^{kt} && \text{for } t \geqslant 0, \\ |y^{(\nu)}(t)| &< M\delta^{n-\nu} && \text{for } -\delta \leqslant t < 0, \end{aligned} \tag{2}$$

where k, $M \geq 0$, and $p \geq 0$ are constants. Then Eq. (1) has a unique solution y(t) which can be represented as the sum of elementary solutions:

$$y(t) = \sum_{i=1}^{\infty} P_i(t) e^{\lambda_i t}. \tag{3}$$

Here the $P_i(t)$ are polynomial with arbitrary constant coefficients of degree one less than that of the root λ_i and the summation is taken over all roots of the characteristic equation

$$\lambda^n + \sum_{v=1}^{n} [a_v \lambda^{n-v} + b_v \lambda^{n-v} e^{-\tau_v \lambda}] = 0. \tag{4}$$

Further, if in formula (3) we sum only over a finite number of roots λ_j of the characteristic Eq. (4), including all roots whose real part is $\geq k$, then the remainder of the series is of the order

$$M\delta^{n-v}\left(1+\frac{t}{\delta}\right)^p e^{kt}. \tag{5}$$

Results of theorem 1. Let $f(t) = 0$. Then, if in the formula corresponding to (3) the summation is carried out only over a finite number of roots λ_j ($j = 1,2,\ldots,m$) of the characteristic Eq. (4), then the remainder is asymptotic with respect to t and approaches the terms or terms in the neglected part of the series whose exponents have the largest real part.

Theorem 2. If the differential-difference equation is an equation with a retarded argument then there exist not more than a finite number of roots of the characteristic equation whose real parts exceed any previously-assigned number.

On the basis of the cited theorems we can affirm that if all the roots of the characteristic Eq. (4) have negative real parts and the summation in (3) is taken over a finite number of roots λ_j ($j = 1,2,\ldots,m$) of the characteristic Eq. (4) then the remainder of the series (5) consists of rapidly diminishing terms. We know that the characteristic Eq. (4) has so-called non-asymptotic and asymptotic roots. We call the non-asymptotic roots of the quasipolynomial those which are located near the axis of coordinates of the complex plane. In addition to the finite number of non-asymptotic roots the characteristic quasipolynomial has a "chain" of more or less evenly distributed roots which continue on to infinity. These roots are called asymptotic roots. Thus the cited theorems permit us to evaluate the influence of the neglected roots upon the solution y(t). If the roots have negative real parts then the remainder of the series consists of rapidly diminishing terms which approach zero more rapidly than the terms corresponding to the non-asymptotic roots. The formula for determining the asymptotic roots of the characteristic equation, in the case where a more precise determination of the transients is required, is contained in [2]. With the aid of this formula we can always calculate the effect of the neglected terms which influence the transient only at the very beginning of the oscillatory process. The theorems cited can serve as the basis of the approximate method of studying the transient processes in stable ACSF which was used in [1].

2. The Phenomenon of "Heredity" in Post-Action Systems

Processes involving post-action are usually called dynamic processes; they are observed in automatic control systems with delays. By delays in ACS we mean the time interval (which in principle always exists) which the system requires to react to the input pulse (the reaction time). Post-action automatic control systems include systems with delays and systems with distributed parameters which are described by means of differential-difference equations and partial differential equations. However, quite often in the case of partial differential equations we obtain after linearization of the differential equations a system of differential-difference equations [4]. Post-action systems are characterized by the fact that the speed of the process is determined by the preceding state of the system. We thus encounter the phenomenon of "heredity;" we will show later that the question of the role played by the initial functions is closely associated with this phenomenon. The question of the role of "heredity" in systems with delays was investigated in [5] for the case of small delays. It was shown in [5] that for some types of equations with sufficiently small delays there is a relaxation of the "heredity" i.e., of the preceding state of the system. For example, for equation

$$\dot{y}(t) + by(t-\tau) = 0 \tag{6}$$

there is a weakening of the influence of "heredity" with time (see [5], page 732) for the condition

$$\tau < \frac{1}{|b|e}. \tag{7}$$

There is a weakening with time of the influence of "heredity" only for those ACSF for which, generally speaking, it is not the delay itself which is smaller, but the product of the modulus of the coefficient of the delay term and the delay time.

For example, for Eq. (6) the condition

$$\tau |b| < \frac{1}{e}. \tag{7a}$$

must be fulfilled.

However we can easily show that for many ACSF the inequality similar to (7a) which guarantees the relaxation of "heredity" is often unsatisfied. In particular this occurs in ACSF with optimum parameters which are described by first-order differential equations with retarded variables.

Let us consider the simplest example which fulfills the optimum aperiodicity criterion [1]; here the relation between the parameters which assures optimalization from the point of view of the stability of the regulation process * has the form

$$\tau be - 1 = 0. \tag{8}$$

This condition is determined by the optimum value of the delay,

$$\tau_{opt} = \frac{1}{be}. \tag{9}$$

When this condition is fulfilled the characteristic equation

$$\lambda + be^{-\lambda\tau} = 0, \tag{10}$$

has, as is easily seen, two equal negative roots

$$\lambda_{1,2} = -\frac{1}{\tau}$$

The solution corresponding to these roots, for the condition $y(t) = \varphi(t)$ over the interval $[-\tau, 0]$ and for $y(0) = y_0$, found by the operational method, is of the form

$$y(t) = \left[\psi(s) - \left(y_0 - \frac{1}{\tau}\psi(s)\right)t\right]e^{-\frac{1}{\tau}t}, \tag{11}$$

where

$$\psi(s) = \int_{-\tau}^{0} \varphi(t)\, e^{-st} dt.$$

For the case where the usual initial conditions are given, without taking the "heredity" into account, $\varphi(t) \equiv 0$ for $t_0 - \tau \le t \le t_0$, $y(t_0) = y_e$, we can easily see that we will have only a partial solution of the form

$$y(t) = y_0 t e^{-\frac{1}{\tau}t}. \tag{12}$$

This solution cannot completely characterize the transient which occurs in the case where the initial function $\varphi(t)$ is given. It follows from the theorems cited above that relaxation occurs only for the elementary solution corresponding to the asymptotic roots of the characteristic equation.

I wish to express my indebtedness to B. V. Shirokorad for his valuable advice which was of great assistance to me in writing this article.

* In this case there is practically no over-regulation of the transient and the damping time is minimal.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

METHOD OF SOLUTION OF MULTIPLE-LOOP SAMPLED DATA SYSTEM EQUATIONS*

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A method is suggested for finding the Laplace transforms and the discrete z-transforms in a sampled data system having an arbitrary structure and a periodic program of pulsed element closings. The initial system is transformed into an equivalent synchronous system and is described in transform theory by a complex of nodal equations. The nodal equations are solved by means of definite rules.

1. Introduction

The simultaneous existence in pulsed systems of continuous and discrete signals makes it very difficult for us to obtain the necessary transforms. In order to obtain the discrete Laplace transform at the output of a single-loop pulsed system G. R. Stibitz and C. E. Shannon [1] introduced several severe limitations upon the transients in the forward and feedback points. An analogous problem for single-loop systems, where no additional limitations were imposed, was solved by Ya. Z. Tsypkin [2] by applying the external action to the output of the pulsed elements (PE). Later D. Ragazzini and L. Zadeh [3] compiled a table of transforms of the output signals for typical single-loop systems containing one or several PE. Ya. Z. Tsypkin [4] obtained a formula for the discrete transform of the signal at the output of a single-loop multiple circuit system with several nonsynchronous PE. In order to find the a-transform at the output of a single-loop system with several PE where the closure period was an integral ratio G. Kranc [5] made use of the structural transformation to reduce the program of closed and open intervals of the PE to a single period. However the solution of the equations obtained for the multiple-loop system was obtained in a very complex manner, by expanding the transforms in infinite series using a special formula. The method of writing the equations for a multiple loop system with several nonsynchronous PE is indicated by Ya. Z. Tsypkin [6]. It is assumed that by using open loop branches at the inputs to the PE the initial system becomes an open pulsed system with a common continuous part and an equal number of inputs and outputs; therefore we consider that the external action is referred to the inputs of the PE.

In the present paper we consider the problem of determining, by algebraic means, the z-transforms for a sampled-data system of arbitrary structure with many PE each of which opens and closes according to an individual program. There is a general repetition rate which is uniform for the system.

The structure of the system is determined by the number of loops which interconnect, in a definite manner, the various branches of the direct (forward) path; each branch may be paralleled by its own feedback loop. In each loop there is an algebraic addition of the signals which arise from within the system or from other branches of the system; the resulting signal is transmitted to all the branches leaving the point [7]. The branches of the linear pulsed system can contain amplifiers, continuous filters, delay elements, PE, discrete filters and devices for converting discrete signals into continuous ones.

2. Pulsed System Synchronization

Let T_r be the repetition rate of the program of closures and openings of the PE of the system. The problem of finding the transforms of the applied signals becomes an algebraic problem after the transformation of the original sampled data system into an equivalent system where all the PE close synchronously and the repetition period is T_r.

* Presented May 27, 1959 at the Seminar on Automatic Regulation Theory which was presided over by Academician B. N. Petrov. The Seminar was sponsored by the Institute of Automation and Remote Control of the Academy of Sciences of the USSR.

Let us replace each programmed closure of a pulsed element by a system of parallel branches. Each branch models one closure in the program and consists of lead elements in series, PE with period T_r and delay elements (elements which introduce a lag); the lead and the lag are numerically equal and are calculated from the start of the period [5].

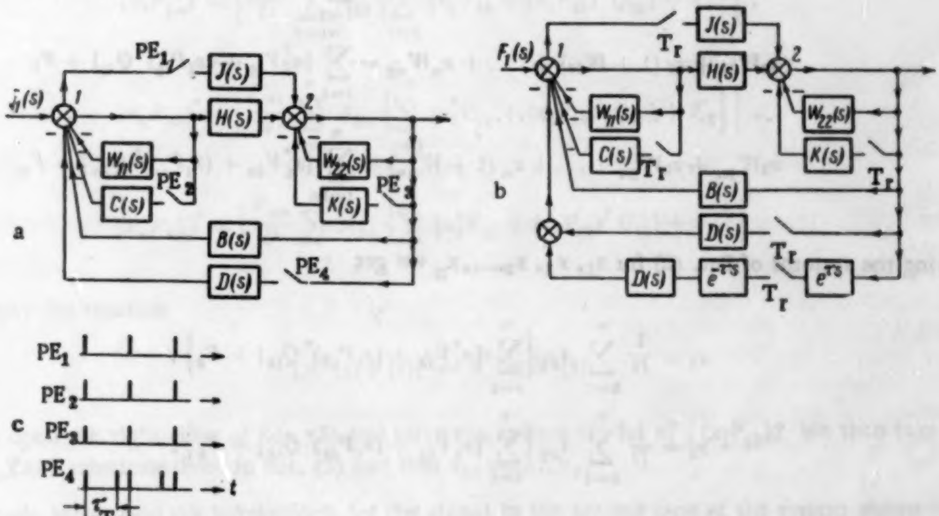

Fig. 1.

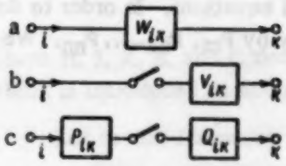

Fig. 2.

As an example let us synchronize the system shown in Fig. 1a where the PE are closed in accordance with the sequence diagram (program) shown in Fig. 1b. Let us replace PE_4 of Fig. 1a by two branches according to the rule indicated above; we then get the synchronous system shown in Fig. 1b where all the PE are closed in synchronism. The repetition period is given by T_r.

We can extend our concept of the programming of the operation of the PE if we assume that the closure of the PE immediately precedes the programmed switching of the PE output to another shaping device or discrete filter. In that case each branch of the equivalent system that models the sequential closure of the given PE is similarly supplied by the corresponding shaping device.

3. Method of Solution of the Loop Equations of the Synchronous Sampled Data System

We will use the following designations: x_k = the Laplace transform of the signal in the k-th loop, x_k^* = the z-transform of the signal in the k-th loop which has a repetition rate of T_r, F_k = Laplace transform of the external signal applied to the k-th loop, W_{ik} = the transfer function of the branch connecting the i-th and k-th loops.

There are, in a synchronous pulsed system, three basic types of branches (Fig. 2): a) branches which do not contain any PE; b) branches which contain only an amplifier between the initial loop and the PE*; c) branches which contain other linear elements in addition to the amplifier between the initial loop and the PE.

Therefore, for a synchronous pulsed system with n-loops we can write the equation for the k-th loop in the form

$$x_k = \sum_{i=1}^{n} [x_i W_{ik} + x_i^* V_{ik} + (x_i P_{ik})^* Q_{ik}] + F_k, \tag{1}$$

where V_{ik}, P_{ik}, and Q_{ik} are transfer functions of the portions of the branches as shown in Fig. 2.

* The coefficient of amplification of the initial part of the branch (Fig. 2b) takes on a unique form in order to simplify the succeeding equations. This does not destroy the generality of the procedure.

Writing the system of equations for all the loops in the system, and transferring the expressions of the type $x_i W_{ik}$ to the left side, we get

$$
\begin{aligned}
x_1(1+W_{11})+x_2W_{21}+\ldots+x_nW_{n1} &= \sum_{i=1}^{n}\left[x_i^* V_{i1}+(x_iP_{i1})^* Q_{i1}\right]+F_1,\\
x_1W_{12}+x_2(1+W_{22})+\ldots+x_nW_{n2} &= \sum_{i=1}^{n}\left[x_i^* V_{i2}+(x_iP_{i2})^* Q_{i2}\right]+F_2,\\
&\ldots\ldots\ldots\ldots\\
x_1W_{1n}+x_2W_{2n}+\ldots+x_n(1+W_{nn}) &= \sum_{i=1}^{n}\left[x_i^* V_{in}+(x_iP_{in})^* Q_{in}\right]+F_n.
\end{aligned}
\tag{2}
$$

Solving the systems of Eqs. (2) for $x_1, x_2, x_3, \ldots, x_n$ we get

$$
\begin{aligned}
x_1 &= \frac{1}{D}\sum_{k=1}^{n} A_{k1}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\},\\
x_2 &= \frac{1}{D}\sum_{k=1}^{n} A_{k2}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\},\\
&\ldots\ldots\ldots\ldots\\
x_n &= \frac{1}{D}\sum_{k=1}^{n} A_{kn}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\},
\end{aligned}
\tag{3}
$$

where $A_{k1}, A_{k2}, \ldots, A_{kn}$ are the minors of the determinant D of the system (1).

We will now supplement the system (3) with the requisite number of differential equations. In order to do this we will multiply each equation of the system (3), for example for x_n, successively by $P_{n1}, P_{n2}, \ldots, P_{nn}$. We then obtain the equations

$$
\begin{aligned}
x_1P_{11} &= \frac{P_{11}}{D}\sum_{k=1}^{n} A_{k1}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\},\\
&\ldots\ldots\\
x_1P_{1n} &= \frac{P_{1n}}{D}\sum_{k=1}^{n} A_{k1}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\},\\
&\ldots\ldots\ldots\ldots\\
x_nP_{n1} &= \frac{P_{n1}}{D}\sum_{k=1}^{n} A_{kn}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\},\\
&\ldots\ldots\\
x_nP_{nn} &= \frac{P_{nn}}{D}\sum_{k=1}^{n} A_{kn}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\}.
\end{aligned}
\tag{4}
$$

Obtaining the z-transforms of the equations in systems (3) and (4) we get a system of n(n + 1) equations for n(n + 1) unknowns of the form x_i^*, (x_iP_{ik}):

$$
\begin{aligned}
x_1^* &= \left|\frac{1}{D}\sum_{k=1}^{n} A_{k1}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\}\right|^*,\\
&\ldots\ldots\ldots\ldots\\
x_n^* &= \left|\frac{1}{D}\sum_{k=1}^{n} A_{kn}\left\{\sum_{i=1}^{n}\left[x_i^* V_{ik}+(x_iP_{ik})^* Q_{ik}\right]+F_k\right\}\right|^*,
\end{aligned}
\tag{5}
$$

$$(x_1P_{11})^* = \left|\frac{P_{11}}{D}\sum_{k=1}^{n} A_{k1}\left\{\sum_{i=1}^{n}\left[x_i^*V_{ik} + (x_iP_{ik})^*Q_{ik}\right] + F_k\right\}\right|^*,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(x_1P_{1n})^* = \left|\frac{P_{1n}}{D}\sum_{k=1}^{n} A_{k1}\left\{\sum_{i=1}^{n}\left[x_i^*V_{ik} + (x_iP_{ik})^*Q_{ik}\right] + F_k\right\}\right|^*,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(x_nP_{n1})^* = \left|\frac{P_{n1}}{D}\sum_{k=1}^{n} A_{kn}\left\{\sum_{i=1}^{n}\left[x_i^*V_{ik} + (x_iP_{ik})^*Q_{ik}\right] + F_k\right\}\right|^*,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(x_nP_{nn})^* = \left|\frac{P_{nn}}{D}\sum_{k=1}^{n} A_{kn}\left\{\sum_{i=1}^{n}\left[x_i^*V_{ik} + (x_iP_{ik})^*Q_{ik}\right] + F_k\right\}\right|^*.$$

We now apply the relation

$$[A^*(z)\,B(z)]^* = A^*(z)\,B^*(z), \tag{6}$$

derived in [3]; we open the right sides of Eqs. (5) and solve the system (5) for x_i^*, $(x_iP_{ik})^*$. We then take the expressions for x_i^* $(x_iP_{ik})^*$ and substitute them in Eqs. (3) and find $x_1, x_2, \ldots, x_n$.

As an example let us find the z-transform for the signal in the second loop of the system shown in Fig. 1a; the equivalent synchronized system is represented by Fig. 1c.

For the latter system we write the equations of loops 1 and 2.

$$\begin{aligned} x_1 &= -x_1W_{11} - x_1^*C - x_2B - x_2^*D - (x_2e^{\tau s})^*De^{-\tau s} + F_1, \\ x_2 &= x_1H - x_1^*J - x_2W_{22} - x_2^*K, \end{aligned} \tag{7}$$

where H, J, K, B, and C denote the transfer functions of the branches (Fig. 1c) and τ is the value of the lag or lead which is introduced in accordance with Fig. 1b.

Let us now introduce the abbreviated designations

$$A = 1 + W_{11}, \qquad P = e^{\tau s}, \qquad Q = De^{-\tau s}, \qquad I = 1 + W_{22}. \tag{8}$$

The system of Eqs. (7) may now be written in the form

$$x_1A + x_2B = -x_1^*C - x_2^*D - (x_2P^*)\,Q + F_1, \qquad -x_1H + x_2I = x_1^*J - x_2^*K. \tag{9}$$

Solving (9) for x_1 and x_2 we obtain the first two equations of system (10). Multiplying the second of these equations by P, we obtain the third equation and thus system (10) becomes complete:

$$\begin{aligned} x_1 &= -x_1^*\frac{BJ + CI}{AI - BH} + x_2^*\frac{BK - DI}{AI - BH} - (x_2P)^*\frac{QI}{AI - BH} + \frac{F_1I}{AI - BH}, \\ x_1 &= x_1^*\frac{AJ + CH}{AI - BH} - x_2^*\frac{AK - DH}{AI - BH} + (x_2P)^*\frac{HQ}{AI - BH} + \frac{F_1I}{AI - BH}, \\ x_2P &= x_1^*\frac{(AJ + CH)\,P}{AI - BH} - x_2^*\frac{(AK - DH)\,P}{AI - BH} + (x_2P)^*\frac{HPQ}{AI - BH} + \frac{F_1HP}{AI - BH}. \end{aligned} \tag{10}$$

Let us now introduce the abbreviated designations, remembering that PQ = D:

$$\begin{gathered} a = 1 + \left(\frac{BJ + CI}{AI - BH}\right)^*, \quad b = \left(\frac{BK - DI}{AI - BH}\right)^*, \quad c = \left(\frac{QI}{AI - BH}\right)^*, \quad d = \left(\frac{F_1I}{AI - BH}\right)^*, \\ e = \left(\frac{AJ + CH}{AI - BH}\right)^*, \quad f = 1 + \left(\frac{AK - DH}{AI - BH}\right)^*, \quad g = \left(\frac{HQ}{AI - BH}\right)^*, \quad h = \left(\frac{F_1I}{AI - BH}\right)^*, \\ i = \left[\frac{(AJ + CH)\,P}{AI - BH}\right]^*, \quad l = \left[\frac{(AK - DH)\,P}{AI - BH}\right]^*, \quad k = 1 - \left(\frac{HD}{AI - BH}\right)^*, \quad m = \left(\frac{F_1HP}{AI - BH}\right)^*; \end{gathered} \tag{11}$$

obtaining the z-transform of Eqs. (10):

$$\begin{aligned} x_1^* a - x_2^* b + (x_2 P)^* c &= d, \\ -x_1^* e + x_2^* f - (x_2 P)^* g &= h, \\ -x_1^* i + x_2^* j + (x_2 P)^* k &= m. \end{aligned} \tag{12}$$

From Eqs. (12) we obtain the z-transform of the output of the sampled data system (Fig. 1a):

$$x_2^* = \frac{ahk + agm + dek + dgi - cem + chi}{afk + agj - bek - bgi - cej + cfi}. \tag{13}$$

The letters in (13) are defined by the transfer functions of the system in agreement with (8)-(11) (Fig. 1a).

If we need to obtain the Laplace transform of the signal in loop 2 (Fig. 1a), then we must find the expressions for x_1^* and $(x_2P)^*$ from the system of Eqs. (12). The latter, and also expression (13) for x_2^*, are then substituted in the first of Eqs. (10).

In conclusion we wish to note that in the last few years papers [10]-[12] have appeared in the foreign press. They are devoted to the solution of the problem of finding the transform of the signal in multiple-loop pulsed systems by means of the basic methods of topological theory of continuous systems in a manner similar to the one that was worked out by S. Mason [8, 9]. Let us stop with the latest and most voluminous paper by G. Lendaris and E. Jury [12], which appeared after the completion of this paper.

For multiple-loop pulsed synchronous systems, equations for the transforms of the output signals were obtained heuristically. The authors indicate that the accuracy of their results remains unproved. The authors have also not made clear the class of systems to which these results apply unless they may be generally applied to all systems.

In the part entitled "Discussion," G. Lendaris introduces (as a new method) a method of solving sampled data systems that is entirely analogous to that which is noted in the introduction to Ya. Z. Tsypkin's method [6]. Also G. Lendaris' method is less general since he assumes a synchronous pulsed system.

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A REVIEW OF THE BOOK "ADAPTIVE CONTROL PROCESSES — A GUIDED TOUR" BY RICHARD BELLMAN

(Princeton University Press, 1961)

A. M. Letov

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Original article submitted

The expression "adaptive processes" appeared for the first time some six or seven years ago; it is most likely that it originally came into being in connection with the announcement by Norbert Wiener of his main cybernetic tenets, namely that the laws of control are identical whether they apply to machines or to living organisms. It is possible under certain conditions to anticipate human reaction, or to predict, given a goal, what path should be followed in order to attain it.

It is thus possible, given sufficient time, to consider for example most carefully some feasible decisions and, by noting their merits and demerits, to choose the optimal one. Such analysis, however, is often out of the question. A skier, for instance, racing down the slope, cannot stop to consider the "pros" and "cons" of all the routes as there is simply no time to do it. In a case like that all he can do is to trust his intuition and choose the track (within the permissible limits) and the speed which seem to be most suitable. It is a well-known fact that in a race like that only those will be successful who have managed to develop during their training and their tests a powerful insight and ability to adapt themselves rapidly to actual conditions prevailing in a particular contest.

Should the skier fall down, this would be regarded as "sheer bad luck" caused by conditions beyond his control and outside his adaptive ability operating in normal and usual conditions. Such reasoning will help us to understand what causes one skier to win and another to lose, especially as this is all that is expected of us when watching these most interesting sporting events.

It is a different matter, however, when by making use of Norbert Wiener's concepts we try to describe in precise mathematical terms:

a) the object under control

b) its adaptive capacity (or ability)

c) the goal to be attained

d) the algorithm and the way of reaching the goal.

Here the difficulty lies in the fact that the mathematician has to state everything in precise and unambiguous terms. A skier has no need of accurate definitions. Nature has equipped him with an adaptive algorithm; it will be of no advantage to him to know its precise formulation since he can perform his tasks very well without knowing it. A theoretician, however, will not be satisfied with just reaching the goal. He will want to know the actual process of reaching it and also whether it can be replicated at will under given conditions. We shall see in the sequel that more than mathematics is needed here; the mathematician must familiarize himself with physiology and in particular the physiology of the brain.

The book under review is, as far as we know, the first in which an attempt has been made to express adaptive control processes from a mathematical angle in an abstract and precise manner; such processes may arise in a priori or not a priori (game-theoretical) situations and with various types of regulation.

A pedantic reader will probably find some weak points in the book; he could, for example, say that it is more of a rough sketch than a full portrait. Personally, I consider the book as the first endeavor to give an accurate exposition of an investigation method which until now has not been sufficiently understood — and I am fascinated by this "portrait."

We know Richard Bellman as the author of a new method of analysis and synthesis of control processes, namely of dynamic programming. The present book is a version,edited by the author,of a series of lectures delivered by him on the theory of adaptive processes; the lectures aroused interest in wide circles in his country — among the scientists as well as engineers, and the personnel of aircraft companies in particular. The book may be regarded as a sequel to his previous book "Dynamic Programming" translated into Russian and published in this country in 1960 with N. N. Vorob'ev as editor. The new book can be recommended without any hesitation to all those who wish to study modern mathematical methods of control processes in whatever field of application.

The aims of the study are formulated in chapter I. The author is interested in physical systems which can be described by means of ordinary differential equations or by integro-differential equations, with hereditary influence also sometimes taken into account. Here the "criteria of performance" are introduced as criteria of optimality. The author formulates the so-termed classical and nonclassical approach to the calculus of variations in the case when one can distinguish the control function in the system. In one case a variational problem with a nonanalytic functional as its performance criterion is pointed out by the author.

Reviewing various problems, the author remarks that though they are within the domain of the classical calculus of variations, it is nevertheless preferable to apply to them different mathematical tools in order to arrive at solutions. What he has in mind, judging by the subsequent chapters, is his dynamic programming method.

The second chapter starts with the author emphasizing that a differential equation

$$\dot{x} = g(x), \qquad x(0) = c,$$

when it has a unique solution for any given $\underline{c}$, determines the transformation group

$$x = x(t, c)$$

of the space of $\{x\}$. This idea used by Poincare, Lyapunov, Andronov, Pontryagin and others, is introduced in connection with dynamic programming. The derivation of functional equations based on this theory is illustrated by an example of the vertical motion of a material point. The forming of functional equations describing multistage process is given in chapter III. In chapter IV the methods are described of solving classical or nonclassical variational problems by means of dynamic programming. Continuous as well as discrete processes are considered. For classical problems it is shown that the functional equations prove equivalent to Euler equations. A functional equation is derived for the case of a nonanalytic functional of the Min Max $|x|$ type.

The difficulties encountered when solving functional equations in an analytic form forced the author to evolve four stages of a computational procedure for the numerical solution of dynamic programming problems. These procedures form the contents of chapter V. Variational problems with integral constraints of the form

$$\int_0^T H(uv)\,dt = k$$

have recently been given a great deal of attention.

This type of constraint is important when the value $\underline{k}$ characterizes the quantity of available energy. The author shows that dynamic programming methods are applicable in these cases if one uses Lagrange multipliers. This is expounded in chapter VI.

In chapter VII a description is given of solving two-point boundary value problems. This kind of problem arises in control processes when an optimum control program must be established, in other words, an optimal free motion (in the Lyapunov sense) of the system takes place when under control.

Chapter VIII is the last one as far as the deterministic control processes are concerned. First the definition is given of the term "sequential machines;" they signify mathematical systems consisting of

a) vector I $(I_1, I_2, \ldots, I_n)$, called input;

b) vector Q $(Q_1, Q_2, \ldots, Q_n)$; called output;

c) vector x $(x_1, x_2, \ldots, x_n)$; called state of the system.

The vectors are interrelated by means of the following rules: 1)The present state x of the system depends only upon the past state and the present input; 2) the present output Q of the system depends only upon the past state and the past input.

The basic problem as regards the sequential machines is as follows: to determine the current state of the machine from the information provided by the rules,assuming this state to be constant until an input I is applied. In the same chapter, dynamic programming methods are further developed and subsequently used in coin-weighing problems and also applied to logical systems.

Chapter IX begins with an introduction of the concepts of uncertainty, of cause and effect, of probability of event, and finally of random processes.

A special effort is made to elucidate the concept of causality and of a Markov process; the Chapman-Kolmogorov equations are derived. These determine,in an analytic form,successive changes of the system in agreement with generalized causality. The equations are subsequently used to obtain the probability distribution of the subsequent outcomes of the random variable $\underline{z}$. In particular, the behavior is studied of a multivibrator governed by the Van der Pol equation with a random forcing function.

Chapter IX serves as an introduction to chapters X and XI in which random processes and discrete Markov processes in particular are considered. The optimization problem is discussed and again dynamic programming is used to obtain its solution. As many of these problems can be reduced to nonlinear differential or differential-difference equations, certain approximate methods called "quasilinearization" are investigated in chapter XII. The approximations are formed in the "policy space," these approximations having the monotonicity property.

In chapter XIII an attempt is made to give a mathematical model of a living organism sequentially adapting itself to its environment according to the received information. These so-called "stochastic learning models" were studied by Bush and Mosteller and also by Flood and by Ashby to whom reference is made by the author. The process of adaptation to environment is approached as a multistage process which selects a particular alternative A_i from the possible alternative actions $A_1, \ldots, A_n$; it is assumed that the model possesses at the initial stage the knowledge of the probabilities $p_1, p_2, \ldots, p_n$ of the alternatives.

In chapter XIV the behavior is studied of a system under control,ruled by recurrence relation

$$x_{n+1} = g(x_n, y_n, r_n), \qquad x(0) = c,$$

where x_n is the vector representing the state of the system at the discrete time $\underline{n}$, y_n represents the control vector, r_n is an independent stochastic vector representing the applied force. As distinct from stochastic processes in chapter X, the probability distribution function of the vector r_n is not in this context assumed to be known. Now, if a maximum or a minimum of a given functional is to be attained, a situation will arise which will subsequently be referred to as a "game against nature." An exposition of the Borel-Neumann theory of games is given in this connection as a multistage process with dynamic programming terminology,which is then applied to the study of control processes. The pursuit of the object Q by the object P is investigated in particular, the following points being emphasized:

1) minimum capture time T;

2) minimum miss distance D;

3) minimum miss distance D within a given time T.

In chapter XV, the author outlines an approach to the concept of adaptive processes. In the foregoing chapters, he says, we have been able to judge accurately either a deterministic state or the probability of such a state within a system of stochastic nature. What should one do in cases when the roles of transition from the initial to the final state remain unknown? This lack of knowledge is classified by the author under three headings; he then goes on to say that the adaptive process presupposes some learning of the unknown. The author perceives the possibility of using such learning in control devices, by the storing of experimental results. The experiments should accumulate in a control device. The latter should be given the possibility of analyzing the experience and improving its performance. Such a controlling device should, of course, contain a "thinking machine." This machine, however, will operate in accordance with an algorithm prepared in advance, as mentioned by the author, but it is still too early to compare it with the human brain. We are still too ignorant of how the brain works,even to start contemplating the construction of a thinking machine. If we could combine the memory and the capability of the human brain

with the accuracy of a modern computer, we would then have an apparatus with which to tackle the main problems of this world.

However, the author makes an attempt in chapter XVI to formulate precisely on an axiomatic basis the concept of optimal policy in adaptive processes. He first introduces the concept of information pattern representing the information about the past history of the process retained in order to guide its behavior in the near future. Having formulated the basic hypotheses, the author develops a method for obtaining functional equations of dynamic programming with the aid of which he is able to describe the adaptive process of a system in general and a controlled one in particular.

To provide a simple illustration of an adaptive process, the random variables r_n are taken such that they will assume the values $r_n = 1$ with probability $\underline{p}$, and the values $r_n = 0$ with probability $1 - p$.

It is assumed that the precise value of $\underline{p}$ is not known; it is assumed, however, that we do possess an "a priori distribution" dG(p) of the quantity $\underline{p}$. We agree also to regard dG(p) as the true probability distribution in the absence of further information and that we possess a definite procedure for modifying dG(p) on the basis of observations made as the process continues.

To this end the Bayes approximation approach can be used. Namely, if it is found that over the preceding m + n observation stages the random variables r_n have taken the value one $\underline{m}$ times and the value zero $\underline{n}$ times, and if the a priori distribution at the beginning of these stages was dG(p), we agree to take the function

$$dG_{mn}(p) = \frac{p^m (1-p)^n \, dG(p)}{\int_0^1 p^m (1-p)^n \, dG(p)}$$

as our new distribution function.

This, of course, is a simple example illustrating a probability adaptive process where the probability is better known after the occurrence of r_n. The problem is easy because the adaptive algorithm has been given beforehand in the form of the Bayes method.

There is one inherent difficulty in designing an adaptive control system; it is due to the fact that the adaptive processes in a living organism are still, with a few exceptions, not understood by engineers or mathematicians. The author observes quite rightly that the brain should be studied first and then a computer designed based on the same principles. As long as we are unable to reproduce in the laboratory a model based on actual physiological processes, no real progress will be made in the design of self-adaptive systems. In the USA some interest has been shown in models of reflexes, in the working of the nervous system and in the adaptive processes of organisms. The reader will probably be disappointed at not finding in the book any elementary examples of self-adaptive systems such as rolling mills, or of some other familiar machines which have been modernized in accordance with Wiener's cybernetics. It seems, however, that the author should not be blamed for this. He only endeavors to foresee forthcoming developments in this field and tries to describe them in general terms by using methods previously developed by him; only the future will tell whether and to what extent his forecast will have proved valuable and useful in practice.

Chapter XVII deals with communication theory. The following problem is, for instance, considered. Let S be the source of the signal, O its observer, let OS represent the communication channel; **x** is a vector representing the pure signal emanating from source S, $\underline{r}$ is the noise mixed in with **x**, x' = F(xr) is the input to the communication channel, **y** is the vector signal transmitted to the output observer O.

We have

$$y = T[F(x, r), x, r].$$

The ensuing analytic problem is as follows. Given that **y** is known from observation. We wish to find out as much as possible about **x**, about the distribution function of $\underline{r}$, the form of F(x, r), and the structure of T. The author remarks that the problem is given in its most general and realistic form.

In this chapter also a method of solving the problem is described, based on dynamic programming; the stochastic process and the process with adaptation are both studied. The latter is treated in precisely the same way as in the preceding chapters.

The last chapter gives an outline of the method of successive approximations in relation to functional equations of dynamic programming. The author has found that the main difficulties when solving functional equations may be overcome by the use of numerical methods, and of the latter the method of successive approximation is given preference. A separate section is devoted to advancing the method in which the approximations are in policy space. These ideas are then applied in the calculus of variations, to automatic control and to the theory of games.

We have so far dealt briefly with the contents of the book. It now remains to mention two facts.

Firstly, the Soviet scientists are very familiar with the work of Richard Bellman, and in particular with his book "Dynamic Programming," translated into Russian. The book under review is written in the same style, giving a brief, laconic but clear exposition. The reviewer is of the opinion that the book should be of considerable scientific interest; the review has as its aim to arouse the interest of the readers of this journal, that is of specialists in automation. The reviewer has tried to render the contents of the book as fully, correctly and clearly as possible.

Secondly, the book is dedicated to A. Lyapunov, H. Poincare, S. Lefschetz and all the other scientists who have devoted their lives to the "improvement of human race." The dedication needs no comment and will be appreciated by all those who have striven to accomplish this great task.

The reviewer is grateful to N. N. Krasovskii for his comments and has endeavoured to incorporate them in the review.

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Soviet Journals Available in Cover-to-Cover Translation

ABBREVIATION	RUSSIAN TITLE	TITLE OF TRANSLATION	PUBLISHER	TRANSLATION BEGAN Vol.	Issue	Year
AÉ	Atomnaya énergiya	Soviet Journal of Atomic Energy	Consultants Bureau	1	1	1956
Akust. zh.	Akusticheskii zhurnal	Soviet Physics – Acoustics	American Institute of Physics	1	1	1955
	Antibiotiki	Antibiotics	Consultants Bureau	4	1	1959
Astr(on). zh(urn).	Astronomicheskii zhurnal	Soviet Astronomy—AJ	American Institute of Physics	34	1	1957
Avto(mat). svarka	Avtomaticheskaya svarka	Automatic Welding	British Welding Research Association (London)		1	1959
	Avtomatika i Telemekhanika	Automation and Remote Control	Instrument Society of America	27	1	1956
	Biofizika	Biophysics	National Institutes of Health*		1	1957
	Biokhimiya	Biochemistry	Consultants Bureau	21	1	1956
Byull. éksp(erim). biol. i med.	Byulleten' éksperimental'noi biologii i meditsiny	Bulletin of Experimental Biology and Medicine	Consultants Bureau	41	1	1959
DAN (SSSR) / Dokl(ady) AN SSSR	Doklady Akademii Nauk SSSR	The translation of this journal is published in sections, as follows:				
	Life Sciences	Doklady Biochemistry Section	American Institute of Biological Sciences	106	1	1956
		Doklady Biological Sciences Sections (Includes: Anatomy, biophysics, cytology, ecology, embryology, endocrinology, evolutionary morphology, genetics, histology, hydrobiology microbiology, morphology, parasitology, physiology, zoology sections)	American Institute of Biological Sciences	112	1	1957
		Doklady Botanical Sciences Sections (Includes: Botany, phytopathology, plant anatomy, plant ecology, plant embryology, plant physiology, plant morphology sections)	American Institute of Biological Sciences	112	1	1957
	Chemical Sciences	Proceedings of the Academy of Sciences of the USSR, Section: Chemical Technology	Consultants Bureau	106	1	1956
		Proceedings of the Academy of Sciences of the USSR, Section: Chemistry	Consultants Bureau	106	1	1956
		Proceedings of the Academy of Sciences of the USSR, Section: Physical Chemistry	Consultants Bureau	112	1	1957
	Earth Sciences	Doklady Earth Sciences Sections (Includes: Geochemistry, geology, geophysics, hydrogeology, mineralogy, paleontology, petrography, permafrost sections)	American Geological Institute	124	1	1959
		Proceedings of the Academy of Sciences of the USSR, Section: Geochemistry	Consultants Bureau	106-123	1 6	1957-1958
		Proceedings of the Academy of Sciences of the USSR, Sections: Geology	Consultants Bureau	106-123	1 6	1957-1958
	Mathematics	Doklady Soviet Mathematics	The American Mathematics Society	131	1	1961
	Physics	Soviet Physics—Doklady (Includes: Aerodynamics, astronomy, crystallography, cybernetics and control theory, electrical engineering, energetics, fluid mechanics, heat engineering, hydraulics, mathematical physics, mechanics, physics, technical physics, theory of elasticity sections)	American Institute of Physics	106	1	1956
		Proceedings of the Academy of Sciences of the USSR, Applied Physics Sections (does not include mathematical physics or physics sections)	Consultants Bureau	106-117	1	1956-1957
Derevoobrabat. prom-st'.	Derevoobrabatyvayushchaya promyshlennost'	Wood Processing Industry	Timber Development Association (London)		9	1959
	Élecktrosvyaz	Telecommunications	Massachusetts Institute of Technology*		1	1957
Éntom(ol). oboz(renie)	Éntomologicheskoe obozrenie	Entomological Review	American Institute of Biological Sciences	38	1	1959
Farmakol. (i) toksikol(ogiya)	Farmakologiya i toksikologiya	Pharmacology and Toxicology	Consultants Bureau	20	1	1957
FMM	Fizika metallov i metallovedenie	Physics of Metals and Metallography	Acta Metallurgica*	5	1	1957
Fiziol. zhurn. SSSR (im. Sechenova)	Fiziologicheskii zhurnal im. I. M. Sechenova	Sechenov Physiological Journal USSR	National Institutes of Health*		1	1957
Fiziol(ogiya) rast.	Fiziologiya rastenii	Plant Physiology	American Institute of Biological Sciences	4	1	1957
	Geokhimiya	Geochemistry	The Geochemical Society		1	1958
FTT	Fizika tverdogo tela	Soviet Physics—Solid State	American Institute of Physics	1	1	1959
Izmerit. tekh(nika)	Izmeritel'naya tekhnika	Measurement Techniques	Instrument Society of America		1	1959
Izv. AN SSSR, O(td). Kh(im). N(auk)	Izvestiya Akademii Nauk SSSR: Otdelenie khimicheskikh nauk	Bulletin of the Academy of Sciences of the USSR: Division of Chemical Sciences	Consultants Bureau		1	1952

continued

Izv. AN SSSR, O(td). T(ekhn). N(auk): Met(all). i top.	(see Met. i top.)					
Izv. AN SSSR Ser. fiz(ich).	Izvestiya Akademii Nauk SSSR: Seriya fizicheskaya	Bulletin of the Academy of Sciences of the USSR: Physical Series	Columbia Technical Translations		1	1954
Izv. AN SSSR Ser. geofiz.	Izvestiya Akademii Nauk SSSR: Seriya geofizicheskaya	Bulletin (Izvestiya) of the Academy of Sciences USSR: Geophysics Series	American Geophysical Union		1	1954
Izv. AN SSSR Ser. geol.	Izvestiya Akademii Nauk SSSR: Seriya geologicheskaya	Izvestiya of the Academy of Sciences of the USSR: Geologic Series	American Geological Institute		1	1958
Kauch. i rez.	Kauchuk i rezina	Soviet Rubber Technology	Research Association of British Rubber Manufacturers	18	1	1959
	Kinetika i kataliz	Kinetics and Catalysis	Consultants Bureau	1	3	1960
	Koks i khimiya	Coke and Chemistry USSR	Coal Tar Research Association (Leeds, England)		1	1958
Kolloidn. zh(urn).	Kolloidnyi zhurnal	Colloid Journal	Consultants Bureau	14	1	1952
	Kristallografiya	Soviet Physics – Crystallography	American Institute of Physics	2	1	1957
Metalov. i term. obrabot. metal	Metallovedenie i termicheskaya obrabotka metallov	Metal Science and Heat Treatment of Metals	Acta Metallurgica	6	1	1958
	Metallurg	Metallurgist	Acta Metallurgica		1	1957
Met. i top.	Metallurgiya i topliva	Russian Metallurgy and Fuels	Eagle Technical Publications		1	1960
Mikrobiol.	Mikrobiologiya	Microbiology	American Institute of Biological Sciences	26	1	1957
OS	Optika i spektroskopiya	Optics and Spectroscopy	American Institute of Physics		6	1959
	Pochvovedenie	Soviet Soil Science	American Institute of Biological Sciences		1	1958
	Priborostroenie	Instrument Construction	British Scientific Instrument Research Association		1	1959
Pribory i tekhn. éks(perimenta)	Pribory i tekhnika éksperimenta	Instruments and Experimental Techniques	Instrument Society of America		1	1957
Prikl. matem. i mekh.	Prikladnaya matematika i mekhanika	Applied Mathematics and Mechanics	American Society of Mechanical Engineers		1	1958
PTÉ	(see Pribory i tekhn. éks.)					
	Problemy Severa	Problems of the North	National Research Council of Canada			
Radiotekh.	Radiotekhnika	Radio Engineering	Massachusetts Institute of Technology*	12	1	1957
Radiotekh. i élektronika	Radiotekhnika i élektronika	Radio Engineering and Electronics	Massachusetts Institute of Technology*	2	1	1957
	Stanki i instrument	Machines and Tooling	Production Engineering Research Assoc.		1	1959
	Stal'	Stal (In English)	Iron and Steel Institute		1	1959
Stek. i keram.	Steklo i keramika	Glass and Ceramics	Consultants Bureau	13	1	1956
Svaroch. proiz-vo	Svarochnoe proizvodstvo	Welding Production	British Welding Research Association		4	1959
Teor. veroyat. i prim.	Teoriya veroyatnostei i ee primenenie	Theory of Probability and Its Applications	Society for Industrial and Applied Mathematics		1	1956
Tsvet. Metally	Tsvetnye metally	Nonferrous Metals	Primary Sources		1	1960
UFN	Uspekhi fizicheskikh Nauk	Soviet Physics – Uspekhi (partial translation)	American Institute of Physics	66	1	1958
UKh	Uspekhi khimii	Russian Chemical Reviews	The Chemical Society (London)		1	1960
UMN	Uspekhi matematicheskikh nauk	Russian Mathematical Surveys	London Mathematical Society	15	1	1960
Usp. fiz. nauk	(see UFN)					
Usp. khim(ii)	(see UKh)					
Usp. matem. nauk	(see UMN)					
Usp. sovr. biol.	Uspekhi sovremennoi biologii	Russian Review of Biology	Oliver and Boyd		48	1959
Vest. mashinostroeniya	Vestnik mashinostroeniya	Russian Engineering Journal	Production Engineering Research Assoc.		4	1959
Vop. gem. i per. krovi	Voprosy gematologii i perelivaniya krovi	Problems of Hematology and Blood Transfusion	National Institutes of Health*		1	1957
Vop. onk.	Voprosy onkologii	Problems of Oncology	National Institutes of Health*		1	1957
Vop. virusol.	Voprosy virusologii	Problems of Virology	National Institutes of Health*		1	1957
Zav(odsk). lab(oratoriya)	Zavodskaya laboratoriya	Industrial Laboratory	Instrument Society of America	25	1	1959
ZhAKh Zh. anal(it). khimii	Zhurnal analiticheskoi khimii	Journal of Analytical Chemistry USSR	Consultants Bureau	7	1	1952
ZhÉTF, Zh. éksperim. i teor. fiz.	Zhurnal éksperimental'noi i theoreticheskoi fiziki	Soviet Physics–JETP	American Institute of Physics	28	1	1955
ZhFKh Zh. fiz. khimii	Zhurnal fizicheskoi khimii	Russian Journal of Physical Chemistry	The Chemical Society (London)		7	1959
ZhMÉI Zh(urn). mikrobiol. épidemiol. i immunobiol.	Zhurnal mikrobiologii, épidemiologii i immunobiologii	Journal of Microbiology, Epidemiology and Immunobiology	National Institutes of Health*		1	1957
ZhNKh, Zh(urn). neorgan(ich). khim(ii)	Zhurnal neorganicheskoi khimii	The Russian Journal of Inorganic Chemistry	The Chemical Society (London)		1	1959
ZhOKh, Zh(urn). obshch(ei) khimii	Zhurnal obshchei khimii	Journal of General Chemistry USSR	Consultants Bureau	19	1	1949
ZhPKh, Zh(urn). prikl. khimii	Zhurnal prikladnoi khimii	Journal of Applied Chemistry USSR	Consultants Bureau	23	1	1950
ZhSKh, Zh(urn). strukt. khimii	Zhurnal strukturnoi khimii	Journal of Structural Chemistry	Consultants Bureau	1	1	1960
ZhTF, Zh(urn). tekhn. fiz.	Zhurnal teknicheskoi fiziki	Soviet Physics–Technical Physics	American Institute of Physics	26	1	1956
Zh(urn). vyssh. nervn. deyat. (im. Pavlova)	Zhurnal vysshei nervnoi deyatel'nosti (im. I. P. Pavlova)	Pavlov Journal of Higher Nervous Activity	National Institutes of Health*		1	1958

*Sponsoring organization. Translation through 1960 issues is a publication of Pergamon Press.

SIGNIFICANCE OF ABBREVIATIONS MOST FREQUENTLY ENCOUNTERED IN SOVIET TECHNICAL PERIODICALS

AN SSSR	*Academy of Sciences, USSR*
FIAN	*Physics Institute, Academy of Sciences USSR*
GITI	*State Scientific and Technical Press*
GITTL	*State Press for Technical and Theoretical Literature*
GOI	*State Optical Institute*
GONTI	*State United Scientific and Technical Press*
Gosénergoizdat	*State Power Press*
Gosfizkhimizdat	*State Physical Chemistry Press*
Goskhimizdat	*State Chemistry Press*
GOST	*All-Union State Standard*
Gostekhizdat	*State Technical Press*
GTTI	*State Technical and Theoretical Press*
IAT	*Institute of Automation and Remote Control*
IF KhI	*Institute of Physical Chemistry Research*
IFP	*Institute of Physical Problems*
IL	*Foreign Literature Press*
IPF	*Institute of Applied Physics*
IPM	*Institute of Applied Mathematics*
IREA	*Institute of Chemical Reagents*
ISN (Izd. Sov. Nauk)	*Soviet Science Press*
IYap	*Institute of Nuclear Studies*
Izd	*Press (publishing house)*
LÉTI	*Leningrad Electrotechnical Institute*
LFTI	*Leningrad Institute of Physics and Technology*
LIM	*Leningrad Institute of Metals*
LITMiO	*Leningrad Institute of Precision Instruments and Optics*
Mashgiz	*State Scientific-Technical Press for Machine Construction Literature*
MGU	*Moscow State University*
Metallurgizdat	*Metallurgy Press*
MOPI	*Moscow Regional Pedagogical Institute*
NIAFIZ	*Scientific Research Association for Physics*
NIFI	*Scientific Research Institute of Physics*
NIIMM	*Scientific Research Institute of Mathematics and Mechanics*
NIKFI	*Scientific Institute of Motion Picture Photography*
NKTM	*People's Commissariat of the Heavy Machinery Industry*
Obrongiz	*State Press of the Defense Industry*
OIYaI	*Joint Institute of Nuclear Studies*
ONTI	*United Scientific and Technical Press*
OTI	*Division of Technical Information*
OTN	*Division of Technical Science*
RIAN	*Radium Institute, Academy of Sciences of the USSR*
SPB	*All-Union Special Planning Office*
Stroiizdat	*Construction Press*
URALFTI	*Ural Institute of Physics and Technology*
TsNIITMASh	*Central Scientific Research Institute of Technology and Machinery*
VNIIM	*All-Union Scientific Research Institute of Metrology*

NOTE: Abbreviations not on this list and not explained in the translation have been transliterated, no further information about their significance being available to us — *Publisher.*

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	United States and Canada	$17.50
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Russian original published by the Ministry of Light Metals, USSR. The articles in this journal relate to instrumentation for analytical chemistry and to physical and mechanical methods of materials research and testing. 1958-1961 issues available.

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